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A symplectic structure for string theory on integrable backgrounds

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Abstract: We define regularised Poisson brackets for the monodromy matrix of classical string theory on $\mathbb{R} \times S^3$. The ambiguities associated with Non-Ultra Locality are resolved using the symmetrisation prescription of Maillet. The resulting brackets lead to an infinite tower of Poisson-commuting conserved charges as expected in an integrable system. The brackets are also used to obtain the correct symplectic structure on the moduli space of finite-gap solutions and to define the corresponding action-angle variables. The canonically-normalised action variables are the filling fractions associated with each cut in the finite-gap construction. Our results are relevant for the leading-order semiclassical quantisation of string theory on $AdS_5 \times S^5$ and lead to integer-valued filling fractions in this context.

Keywords: AdS-CFT Correspondence, Integrable Field Theories, Sigma Models, Differential and Algebraic Geometry.

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1. Introduction

Determining the exact spectrum of free string theory on $AdS_5 \times S^5$ is an important problem whose solution would surely lead to a better understanding of the AdS/CFT correspondence. The discovery of integrability in the classical theory [5] is a good indication that the problem may be tractable. More precisely, the authors of [5] found a Lax formulation of the equations of motion which leads the existence of an infinite tower of conserved charges in the classical worldsheet theory. These charges have subsequently been exploited to construct and classify large families of exact solutions of the classical equations of motion [6 − 8, 1]. However, this does not quite coincide with the standard definition of integrability. Integrability in the standard sense requires not only the existence of a tower of conserved charges but also requires that these charges be "in involution". In other words the conserved charges should Poisson commute with each other. In finite-dimensional systems, this is a necessary condition for Liouville's theorem[1] to hold. More generally, knowledge of the Poisson brackets is necessary for constructing the action-angle variables for the system which play a key role in semiclassical quantisation. In this paper, which builds on our earlier work [1], we will derive the involution condition for classical strings moving on an $\mathbb{R} \times S^3$ submanifold of $AdS_5 \times S^5$ and construct the corresponding action-angle variables.

In classical string theory on $AdS_5 \times S^5$, as well as many other backgrounds which admit a Lax formulation, there is a long-standing problem in determining the Poisson brackets of the conserved charges. As we review below, the problem is due to the presence of *Non-Ultra Local* (NUL) terms in the Poisson brackets of the worldsheet fields which lead to ambiguities in brackets for the charges. In this paper we will present a resolution of this problem based on earlier work by Maillet [20, 18, 19] in the context of two dimensional field theory. In particular, Maillet proposed a prescription for regularising the problematic brackets. In the following we will apply his procedure to the simplest classical subsector of the $AdS_5 \times S^5$ theory which corresponds to bosonic strings moving on an $\mathbb{R} \times S^3$ submanifold of the full geometry. We will show that this prescription leads to a very natural symplectic structure on the space of finite-gap solutions of the string equations of motion constructed in [1]. In particular, we find that this symplectic structure leads to canonically normalised action variables which are exactly equal to the filling fractions discussed in [1]. Our results are relevant for the leading-order semiclassical quantisation of strings moving on an $\mathbb{R} \times S^3$ submanifold of $AdS_5 \times S^5$. In this context, they confirm the expected integer quantisation of the filling fractions discussed in [1]. Our methods should generalise to other sectors of classical strings on $AdS_5 \times S^5$ and also to other integrable backgrounds. In the rest of this introductory section we will outline the main ideas in the paper.

Bosonic strings moving on $\mathbb{R} \times S^3$ are described in static gauge by an SU(2)-valued world-sheet field $g(\sigma, \tau)$ which gives rise to a conserved current $j_\mu(\sigma, \tau) = -g^{-1}\partial_\mu g$. The corresponding action for $g(\sigma, \tau)$ is essentially that of the SU(2) Principal Chiral model,

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \; \frac{1}{2}\mathrm{tr}(j_\mu j^\mu) \tag{1.1}$$

where λ is a dimensionless coupling constant. Physical motions of the string also obey the Virasoro constraint,

$$\frac{1}{2}\mathrm{tr} j_\pm^2 = -\kappa^2 \tag{1.2}$$

where $j_\pm = j_0 \pm j_1$ are the lightcone components of the current and κ is a constant related to the spacetime energy of the string. For many purposes it is convenient to complexify

[1] Liouville's theorem applies to dynamical systems with N degrees of freedom which also have N, globally defined, conserved charges in involution. The theorem guarantees that the equations of motion can be solved by quadratures for arbitrary initial data (see e.g. [9]).

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the model and work with a current j_μ taking values in the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. A solution of the original problem where j_μ is restricted to lie in $\mathfrak{su}(2)$ is then obtained by imposing appropriate reality conditions.

Starting from the action (1.1) it is straightforward to obtain the (equal-τ) Poisson brackets for the components of the current $j_\mu(\sigma)$. Writing the current as $j_0 = j_0^a t^a$, $j_1 = j_1^a t^a$, in terms of SU(2) generators t^a satisfying,

$$[t^a, t^b] = f^{abc} t^c, \quad \mathrm{tr}(t^a t^b) = -\delta^{ab}.$$

the resulting brackets are,

$$\begin{aligned} \left\{ j_1^a(\sigma), j_1^b(\sigma') \right\} &= 0, \\ \frac{\sqrt{\lambda}}{4\pi} \left\{ j_0^a(\sigma), j_1^b(\sigma') \right\} &= -f^{abc} j_1^c(\sigma)\delta(\sigma - \sigma') - \delta^{ab}\delta'(\sigma - \sigma'), \\ \frac{\sqrt{\lambda}}{4\pi} \left\{ j_0^a(\sigma), j_0^b(\sigma') \right\} &= -f^{abc} j_0^c(\sigma)\delta(\sigma - \sigma'). \end{aligned} \tag{1.3}$$

These brackets are usually described as Non-Ultra Local (NUL) reflecting the presence the the derivative $\delta'(\sigma - \sigma')$ in the second bracket. As we now review, the problems related to the NUL nature of these brackets emerge when we consider the corresponding Poisson brackets of the infinite tower of conserved charges of the model. The starting point for constructing these charges is the existence of a one-parameter family of flat currents,

$$J(x) = \frac{1}{1 - x^2}(j - x * j), \tag{1.4}$$

labelled by the complex spectral parameter $x \in \mathbb{C}$. The flatness of $J(x)$, for all values of x, is equivalent to the equations of motion which follow from the action (1.1).

Using the current $J(x)$, we can construct a monodromy matrix,

$$\Omega(x, \sigma, \tau) = P\overleftarrow{\exp} \int_{[\gamma(\sigma,\tau)]} J(x) \ \in \ \mathrm{SL}(2, \mathbb{C}) \tag{1.5}$$

where $\gamma(\sigma, \tau)$ is a non-contractible loop on the string worldsheet based at the point (σ, τ). The flatness of $J(x)$ implies that $\Omega(x)$ undergoes isospectral evolution in the world-sheet coordinates. In other words the eigenvalues of the monodromy matrix are independent of σ and τ. As $\Omega(x)$ takes values in SU(2) when $x \in \mathbb{R}$, it is convenient to parametrise the eigenvalues as,

$$\lambda_\pm = \exp\left(\pm i p(x)\right). \tag{1.6}$$

Here $p(x)$ is a (multi-valued) function of the spectral parameter which is known as the *quasi-momentum.* The Taylor coefficients in the expansion of $p(x)$ then generate an infinite tower of conserved quantities on the worldsheet.

The Poisson bracket for the conserved charges can be deduced from the Poisson bracket $B(x, x') = \{\Omega(x) \overset{\otimes}{,} \Omega(x')\}$ for the monodromy matrix. To calculate $B(x, x')$, we begin by defining a transition matrix between distinct points σ_1 and σ_2 on the string,

$$T(\sigma_1, \sigma_2, x) = P\overleftarrow{\exp} \int_{\sigma_2}^{\sigma_1} d\sigma J_1(\sigma, x).$$

Using the Poisson brackets (1.3) of the current we can calculate the bracket,

$$\Delta^{(1)}(\sigma_1, \sigma_2, \sigma_1', \sigma_2'; x, x') = \{T(\sigma_1, \sigma_2, x) \overset{\otimes}{,} T(\sigma_1', \sigma_2', x')\}$$

This is well defined when the points $\sigma_1, \sigma_2, \sigma_1', \sigma_2'$ are all distinct. However the presence of the distribution $\delta'(\sigma - \sigma')$ on the r.h.s. of the second bracket in (1.3) leads to a finite discontinuity on surfaces where two of the points coincide. To obtain the desired bracket $B(x, x')$ we must take the limit $\sigma_1 \to \sigma_1'$, $\sigma_2 \to \sigma_2'$ and the discontinuity of $\Delta^{(1)}$ on this surface leads to an ambiguous result.

The ambiguity described above is quite mild for the bracket $B(x, x')$ itself, but becomes more serious when one tries to define nested Poisson brackets for a product of monodromy matrices. The ambiguities then result from multiple coincident endpoints of the corresponding transition matrices. To resolve the ambiguity one can introduce an infinitesimal splitting between these coincident endpoints. Fortunately there is a straightforward prescription due to Maillet which seems to provide the unique consistent resolution of the problem. As we review in section 2.4.2, Maillet's prescription involves a total symmetrisation over all possible point-splittings. The prescription preserves the defining properties of the Poisson bracket such as its anti-symmetry, the Leibniz rule and the Jacobi identity. The resulting bracket of two monodromy matrices can be then be written as,

$$\begin{aligned}\left\{\Omega(x) \overset{\otimes}{,} \Omega(x')\right\} =& [r(x, x'), \Omega(x) \otimes \Omega(x')] \\ &+ (\Omega(x) \otimes \mathbf{1})\, s(x, x') \left(\mathbf{1} \otimes \Omega(x')\right) \\ &- \left(\mathbf{1} \otimes \Omega(x')\right) s(x, x') \,(\Omega(x) \otimes \mathbf{1})\,, \end{aligned} \tag{1.7}$$

where,

$$r(x, x') = -\frac{2\pi}{\sqrt{\lambda}} \frac{x^2 + x'^2 - 2x^2 x'^2}{(x - x')(1 - x^2)(1 - x'^2)}, \qquad s(x, x') = -\frac{2\pi}{\sqrt{\lambda}} \frac{x + x'}{(1 - x^2)(1 - x'^2)}. \tag{1.8}$$

Finally, using this relation one may compute the bracket,

$$\left\{\mathrm{tr}\ \Omega(x)^n, \mathrm{tr}\ \Omega(x')^m\right\} = 0. \tag{1.9}$$

As above the eigenvalues of $\Omega(x)$ yield a one-parameter family of conserved charges. The bracket (1.9) therefore implies that the charges corresponding to different values of the spectral parameter x Poisson commute. This is the natural generalisation of the involution condition discussed above for an infinite dimensional system.

The main goal of this paper is to explore the consequence of Maillet's prescription for the finite-gap solutions of the string equations of motion discussed in [1]. Solutions carry the conserved charges Q_L and Q_R associated with the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ isometry group of the target S^3. As in [1] we will focus on solutions of highest weight with respect to both SU(2) factors which have,

$$Q_R = \frac{1}{2i} R\sigma_3, \; Q_L = \frac{1}{2i} L\sigma_3$$

where $\sigma_3 = \mathrm{diag}(1, -1)$ is the third Pauli matrix. The required solutions are characterised by the analytic behaviour of the corresponding quasi-momentum $p(x)$ in the spectral plane.

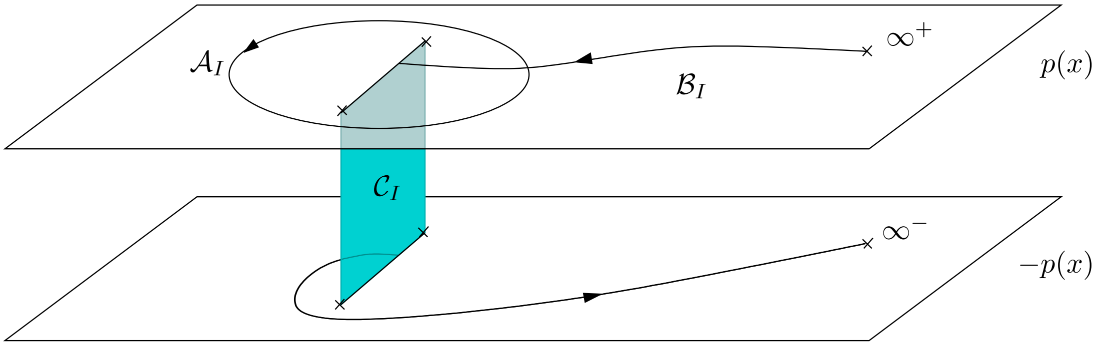

Figure 1: The cycle $\mathcal{A}_I$ and path $\mathcal{B}_I$ for the cut $\mathcal{C}_I$.

The definition (1.6) implies that $p(x)$ need not be single-valued, but can have discontinuities of the form

$$p(x+\epsilon)+p(x-\epsilon)=2\pi n_I, \quad x\in\mathcal{C}_I, \quad n_I\in\mathbb{Z}, \quad I=1,\dots,K. \tag{1.10}$$

across square-root branch cuts $\mathcal{C}_I$ in the x-plane. Finite-gap solutions correspond the case where K, the number of such cuts, is finite. In this case, the resulting double-cover of the x plane defines a hyperelliptic Riemann surface Σ of finite genus $g=K-1$ known as the spectral curve. It is convenient to define a basis of one-cycles on Σ as follows. For $I=1,\dots,K$, the contour $\mathcal{A}_I$ surrounds the cut $\mathcal{C}_I$ on the upper sheet while $\mathcal{B}_I$ runs from the point at infinity on the upper sheet to the same point on the lower sheet via the cut $\mathcal{C}_I$ (see figure 1).

The quasi-momentum $p(x)$ gives rise to a meromorphic differential dp on Σ with periods,

$$\int_{\mathcal{A}_I} dp = 0, \quad \int_{\mathcal{B}_I} dp = 2\pi n_I, \; n_I\in\mathbb{Z}. \tag{1.11}$$

The explicit reconstruction of solutions from the holomorphic data $\{\Sigma, dp\}$ was described in detail in [1] and is also reviewed below in section 3.1. Here we will only summarise the main features. After taking into account the various constraints on the data, Riemann surfaces Σ and differentials dp corresponding to physical solutions are parametrised by K moduli $\mathcal{S}_I$ defined as,

$$\mathcal{S}_I = \frac{1}{2\pi i}\frac{\sqrt{\lambda}}{4\pi}\int_{\mathcal{A}_I}\left(x+\frac{1}{x}\right)dp \tag{1.12}$$

for $I=1,\dots,K$. For real solutions, the moduli are real numbers corresponding to the independent conserved charges of the model carried by the configuration. They are further constrained by the relations,

$$\sum_{I=1}^{K}\mathcal{S}_I = \frac{1}{2}(L-R), \qquad \sum_{I=1}^{K} n_I\mathcal{S}_I = 0. \tag{1.13}$$

The first equality suggests that we should identify $\mathcal{S}_I$ as the amount of angular momentum $J_2=(L-R)/2$ associated with each cut $\mathcal{C}_I$. In the context of the AdS/CFT correspondence these variables correspond to the filling fractions which count the total number of Bethe roots associated with each cut. The second equation in (1.13) corresponds to the constraint that the total worldsheet momentum should vanish.

The moduli $\mathcal{S}_I$ correspond to the conserved quantities of the corresponding string motion or the 'action' variables. On general grounds, we expect that each conserved quantity has a corresponding conjugate variable which is periodic and evolves linearly in time. The extra information required to uniquely specify a solution is just the initial values of these 'angle' variables. In [1], we identified this data with a divisor γ of degree g on Σ and an additional angular variable $\bar{\theta}$ describing the global orientation of the string. Here we will use an equivalent description in terms of a divisor $\hat{\gamma}$ of degree $K = g+1$ on Σ. This in turn uniquely specifies a point $\vec{\mathcal{A}}(\hat{\gamma})$ in the generalised Jacobian $J(\Sigma, \infty^\pm)$ (topologically equivalent to $J(\Sigma) \times \mathbb{C}^*$) via the extended Abel map. The σ and τ-evolution of the solution correspond to the linear motion of this point. Finally to obtain a real solution, the point $\vec{\mathcal{A}}(\hat{\gamma})$ is constrained to lie on the real slice,[2]

$$T^K \simeq \mathrm{Re}\left[J(\Sigma) \times \mathbb{C}^*\right].$$

We define a set of coordinates $\vec{\varphi} = (\varphi_1, \ldots, \varphi_K)$ on the real torus T^K normalised so that $\varphi_I \in [0, 2\pi]$ for $I = 1, \ldots, K$. As these variables evolve linearly in the worldsheet time they correspond to the normalised angle variables of the solution.

The space of finite-gap solutions is a real manifold of dimension $2K$ parametrised by the coordinates $\{S_I, \varphi_I\}_{I=1}^K$, introduced above. The symplectic structure on the infinite dimensional field space of the string defined by the regularised Poisson brackets (1.7) induces a symplectic structure on this manifold. Our main result is an explicit formula for the corresponding symplectic form $\hat{\omega}_{2K}$;

$$\hat{\omega}_{2K} = \sum_{I=1}^{K} \delta\mathcal{S}_I \wedge \delta\varphi_I. \tag{1.14}$$

As the angular variables φ_I each have period 2π variables, the canonically conjugate variables $\mathcal{S}_I$ are the correctly normalised action variables for the problem. The Bohr-Sommerfeld condition for leading-order semiclassical quantisation of the finite-gap solutions therefore simply imposes the integrality of the filling fractions.

The rest of the paper is organised as follows. In section 2, we describe the Hamiltonian formulation of classical string theory on $\mathbb{R} \times S^3$. In particular we derive the Poisson brackets (1.3) and discuss the Maillet regularisation prescription leading to the involution condition (1.9). In section 3, we use the regularised brackets to obtain the symplectic form (1.14) on the space of finite-gap solutions. Along the way, in subsection 3.1 we provide a review of the construction of finite-gap solutions given in [1]. This subsection also contains a new explicit formula for the original σ-model fields corresponding to a genus g finite-gap solution. Throughout this section we emphasise the parallels between finite-gap solutions and the conventional mode expansion for strings in flat space. The

[2]See section 3.1.4 and 3.4.2 for a more precise discussion of the reality conditions.

remainder of section 3 describes the pullback of the symplectic form to the moduli space of finite-gap solutions and the corresponding action-angle variables. Some of the more lengthy calculations are relegated to three appendices.

2. Classical integrability of strings on $\mathbb{R}\times S^3$

2.1 Strings on $\mathbb{R}\times S^3$

The embedding of the string in $\mathbb{R}\times S^3$ is described by the time coordinate $X_0(\sigma,\tau)\in\mathbb{R}$ along with a matrix

$$g(\sigma,\tau)=\begin{pmatrix} X_1+iX_2 & X_3+iX_4 \\ -X_3+iX_4 & X_1-iX_2 \end{pmatrix}\equiv\begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix}\in \mathrm{SU}(2) \tag{2.1}$$

describing the embedding in S^3. In conformal gauge the action can be written in terms of a current $j=-g^{-1}dg$ and the time coordinate X_0 as follows

$$S=-\frac{\sqrt{\lambda}}{4\pi}\int\left[\frac{1}{2}\mathrm{tr}(j\wedge *j)+dX_0\wedge *dX_0\right]. \tag{2.2}$$

The equations of motion that follow from this action are

$$d*j=0,\quad dj-j\wedge j\equiv 0,\qquad d*dX_0=0, \tag{2.3}$$

where the second equation is an identity following from the definition of j. To describe physical motions of the string, the equations of motion (2.3) have to be supplemented by the Virasoro constraints which in conformal gauge read

$$\frac{1}{2}\mathrm{tr}j_\pm^2=-(\partial_\pm X_0)^2. \tag{2.4}$$

where $j_\pm=j_0\pm j_1$ are the components of the current j in the worldsheet light-cone coordinates $\sigma^\pm=\frac{1}{2}(\tau\pm\sigma)=\frac{1}{2}(\sigma^0\pm\sigma^1)$. The equation of motion for X_0 in (2.3) is decoupled from the other fields and hence can be solved separately. In analogy with the flat space case we can then make use of the residual gauge symmetry to impose say static gauge, $X_0=\kappa\tau$, which fixes the τ coordinate and leaves only the possibility of rigid translations in the σ coordinate

$$\sigma\to\sigma+const. \tag{2.5}$$

Thus, working in conformal static gauge, the original gauge invariance of the full string action is completely fixed except for the global transformation (2.5) under which physical states must be invariant. The conserved charges associated with rigid translations of the worldsheet coordinate σ and τ are $\mathcal{P}$ and $\mathcal{E}-\sqrt{\lambda}\kappa^2/2$ where

$$\mathcal{P}=-\frac{\sqrt{\lambda}}{4\pi}\int_0^{2\pi}d\sigma\,\mathrm{tr}\left[j_0j_1\right],\quad \mathcal{E}=-\frac{\sqrt{\lambda}}{4\pi}\int_0^{2\pi}d\sigma\,\frac{1}{2}\mathrm{tr}\left[j_0^2+j_1^2\right]. \tag{2.6}$$

In static gauge, since X_0 is completely specified, only the equations of motion for the current j in (2.3) remain and the Virasoro constraint simplify to

$$\frac{1}{2}\mathrm{tr}j_\pm^2=-\kappa^2. \tag{2.7}$$

It will be convenient to postpone implementing the momentum constraint $\mathcal{P} = 0$ until the very end of the calculation. Hence we split the constraints (2.7) into two parts. The first set of constraints read,

$$\frac{1}{2}\mathrm{tr} j_{\pm}^2 = -\kappa_{\pm}^2, \tag{2.8}$$

where $\kappa_{\pm}$ are constants. The world-sheet momentum and energy (2.6) then become

$$\mathcal{P} = \frac{\sqrt{\lambda}}{4}(\kappa_+^2 - \kappa_-^2), \quad \mathcal{E} = \frac{\sqrt{\lambda}}{4}(\kappa_+^2 + \kappa_-^2). \tag{2.9}$$

The remaining content of the Virasoro constraint (2.7) is the vanishing of the total momentum $\mathcal{P} = 0$ which implies $\kappa_+^2 = \kappa_-^2 = \kappa^2$ and the string mass-shell condition,

$$\mathcal{E} = \frac{\sqrt{\lambda}}{2}\kappa^2 = \frac{\Delta^2}{2\sqrt{\lambda}}. \tag{2.10}$$

The action (2.2) has the following global $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry

$$g \to U_L g U_R, \quad U_L, U_R \in \mathrm{SU}(2),$$

with corresponding Noether charges

$$\mathrm{SU}(2)_R: \quad Q_R = \frac{\sqrt{\lambda}}{4\pi}\int_\gamma *j, \qquad \mathrm{SU}(2)_L: \quad Q_L = \frac{\sqrt{\lambda}}{4\pi}\int_\gamma * \left(gjg^{-1}\right),$$

where γ is any closed curve winding once around the world-sheet. Since these charges are conserved classically, without loss of generality we can restrict attention to 'highest weight' solutions defined by the level set

$$Q_R = \frac{1}{2i}R\sigma_3, \; Q_L = \frac{1}{2i}L\sigma_3, \quad R, L \in \mathbb{R}_+. \tag{2.11}$$

Any other solution with Casimirs $Q_R^2 = R^2, Q_L^2 = L^2$ can be obtained from a 'highest weight' solution by applying to it a combination of $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_L$ transformations. Note that the current j is $\mathrm{SU}(2)_L$ invariant, but transforms under $\mathrm{SU}(2)_R$ by conjugation

$$j \to U_R^{-1} j U_R.$$

2.2 Hamiltonian framework

Starting from the action (2.2) we first derive the Poisson brackets of the system. It will be convenient to choose as our generalised coordinates, the (target-space) time coordinate $q^0(\sigma) = X_0(\sigma)$ and the spatial component of the $\mathrm{SU}(2)_R$ current, $q^a(\sigma) = j_1^a(\sigma)$ for $a = 1, 2, 3$. To proceed further we first choose a particular basis t^a of the Lie algebra $\mathfrak{su}(2)$ with structure constants f^{abc} and normalised such that

$$[t^a, t^b] = f^{abc}t^c, \quad \mathrm{tr}(t^a t^b) = -\delta^{ab}.$$

The action (2.2) then reads

$$S = \frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma \left[\sum_{a=1}^{n} \frac{1}{2}\left[(j_0^a)^2 - (j_1^a)^2\right] - \dot{X}_0^2 + X_0'^2\right].$$

Here the dot and prime denote differentiation with respect to the worldsheet coordinates τ and σ respectively. The conjugate momentum for the time coordinate $X_0(\sigma)$ is given by,

$$\pi^0(\sigma) = \frac{\delta S}{\delta \dot{q}^0(\sigma)} = -\frac{\sqrt{\lambda}}{2\pi}\dot{X}_0(\sigma).$$

By the flatness of j it follows that $\dot{q}^a(\sigma) = \frac{\partial j_0^a}{\partial \sigma} - [j_1, j_0]^a = \nabla_1 j_0^a$, where ∇_1 is the covariant derivative ($\nabla = d - j$) for the connection $j_1 = j_1^a t^a$. The conjugate momentum of $q^a(\sigma)$ is then

$$\begin{aligned}\pi^a(\sigma) &= \frac{\delta S}{\delta \dot{q}^a(\sigma)} = -\frac{\sqrt{\lambda}}{8\pi}\int \sum_{b=1}^{n} \frac{\delta (j_0^b)^2(\sigma')}{\delta \dot{q}^a(\sigma)} d\sigma' d\tau' = -\frac{\sqrt{\lambda}}{4\pi}\int \sum_{b=1}^{n} \frac{\delta j_0^b(\sigma')}{\delta \dot{q}^a(\sigma)} j_0^b(\sigma') d\sigma' d\tau' \\ &= -\frac{\sqrt{\lambda}}{4\pi}\int \sum_{b=1}^{n} \nabla_1^{-1}(\delta_a^b \delta(\sigma-\sigma')\delta(\tau-\tau')) j_0^b(\sigma') d\sigma' d\tau' = \frac{\sqrt{\lambda}}{4\pi}\nabla_1^{-1} j_0^a(\sigma).\end{aligned}$$

In other words, $\frac{\sqrt{\lambda}}{4\pi} j_0^a(\sigma) = \nabla_1 \pi^a(\sigma)$ for $a = 1, 2, 3$. Introducing a new index $A = 0, 1, 2, 3$, the full set of canonical Poisson brackets between the generalised coordinates and their conjugate momenta are,

$$\begin{aligned}\{q^A(\sigma), q^B(\sigma')\} &= \{\pi^A(\sigma), \pi^B(\sigma')\} = 0 \\ \{q^A(\sigma), \pi^B(\sigma')\} &= \delta^{AB}\delta(\sigma-\sigma').\end{aligned}$$

It is convenient to rewrite these Poisson brackets by eliminating three conjugate momenta π_A for $A > 0$, in favour of the current components j_0^a to obtain,

$$\left\{ j_1^a(\sigma), j_1^b(\sigma') \right\} = 0, \tag{2.12a}$$

$$\begin{aligned}\frac{\sqrt{\lambda}}{4\pi}\left\{ j_0^a(\sigma), j_1^b(\sigma') \right\} &= \left\{ \nabla_1 \pi^a(\sigma), q^b(\sigma') \right\} \\ &= -\left\{ q^b(\sigma'), \partial_\sigma \pi^a(\sigma) \right\} + f^{adc}\left\{ q^b(\sigma'), q^d(\sigma)\pi^c(\sigma) \right\} \\ &= -\partial_\sigma \left(\delta^{ab}\delta(\sigma'-\sigma) \right) + f^{adc} q^d(\sigma)\delta^{bc}\delta(\sigma'-\sigma) \\ &= -f^{abc} j_1^c(\sigma)\delta(\sigma-\sigma') - \delta^{ab}\delta'(\sigma-\sigma'),\end{aligned} \tag{2.12b}$$

$$\frac{\sqrt{\lambda}}{4\pi}\left\{ j_0^a(\sigma), j_0^b(\sigma') \right\} = -f^{abc} j_0^c(\sigma)\delta(\sigma-\sigma'). \tag{2.12c}$$

As expected, the $\mathrm{SU}(2)_R$ symmetry is generated by the Noether charge Q_R. Indeed, we find from the last two equations that the Noether charge Q_R acts on the $\mathrm{SU}(2)_R$ current j as follows

$$\{\epsilon \cdot Q_R, j\} = [j, \epsilon] = \delta_\epsilon j, \tag{2.13}$$

where $\epsilon = \epsilon^a t^a \in \mathfrak{su}(2)$ is infinitesimal and $\epsilon \cdot Q_R = \epsilon^a Q_R^a$.

In the Hamiltonian formalism, the dynamics of the string is encoded in the Virasoro constraints,

$$\begin{aligned}\mathcal{H}_\tau &= \sum_{a=1}^{3}\left[(j_0^a)^2+\left(j_1^b\right)^2\right]-\frac{4\pi^2}{\lambda}\left(\pi^0\right)^2-X_0'^2=0,\\ \mathcal{H}_\sigma &= \sum_{a=1}^{3}(j_0^a j_1^a)+\frac{2\pi}{\lambda}\pi_0 X_0'=0.\end{aligned} \tag{2.14}$$

The corresponding Hamiltonian takes the form

$$H=\frac{1}{2\pi}\int_0^{2\pi}d\sigma\left(\mathcal{N}_\tau\mathcal{H}_\tau+\mathcal{N}_\sigma\mathcal{H}_\sigma\right),$$

where $\mathcal{N}_\tau(\sigma)$ and $\mathcal{N}_\sigma(\sigma)$ are Lagrange multipliers for the Virasoro constraints.

As in section 2.1 we will choose to work in static gauge. This corresponds to setting $X_0=\kappa\tau$ and $\pi_0=-\sqrt{\lambda}\kappa/2\pi$ where, as before, κ is related to the string energy as $\kappa=\Delta/\sqrt{\lambda}$. In this gauge, rigid translations of the world-sheet coordinates τ and σ are generated by the Hamiltonian functions,

$$\begin{aligned}H_\tau^{\text{static}} &= \frac{1}{2\pi}\int_0^{2\pi}d\sigma\sum_{a=1}^{3}\left[(j_0^a)^2+\left(j_1^b\right)^2\right],\\ H_\sigma^{\text{static}} &= \frac{1}{2\pi}\int_0^{2\pi}d\sigma\, 2\sum_{a=1}^{3}(j_0^a j_1^a).\end{aligned}$$

The zero momentum components of the Virasoro constraints correspond to the string mass-shell condition $H_\tau^{\text{static}}=\Delta^2/\sqrt{\lambda}$ and the condition that the total world-sheet momentum should vanish: $H_\sigma^{\text{static}}=0$.

Even though the Virasoro constraints are first class by themselves, the static gauge fixing conditions $X_0=\kappa\tau$ and $\pi_0=-\sqrt{\lambda}\kappa/2\pi$ are second class since $\{X_0(\sigma'),\pi(\sigma)\}=\delta(\sigma-\sigma')\neq 0$. This means that to impose all the constraints in the Hamiltonian framework one must work with Dirac brackets instead of Poisson brackets.[3] However, as we argue in appendix D, for the set of action-angle variables which we are concerned with in this paper the Dirac brackets are the same as their Poisson brackets since these variables can be defined in a conformally invariant way.

In the following it will be convenient to think of the infinite-dimensional phase space of the model, denoted $\mathcal{P}^\infty$ as consisting of all configurations $j_0(\sigma),j_1(\sigma)\in\mathfrak{su}(2)$ which obey the Virasoro constraints (2.7). Sometimes we will also consider the complexified phase-space $\mathcal{P}^\infty_{\mathbb{C}}$ with $j_0(\sigma),j_1(\sigma)\in\mathfrak{sl}(2,\mathbb{C})$.

2.3 Conserved charges

The starting point for constructing the infinite tower of conserved charges for the system is to rewrite its equations of motion (2.3) as the flatness condition

$$dJ(x)-J(x)\wedge J(x)=0, \tag{2.15}$$

[3] We are very grateful to Marc Magro for pointing out this issue.

for some family of current $J(x)$ on the world-sheet defined in this case as

$$J(x) = \frac{1}{1-x^2}\left(j - x * j\right).$$

Owing to the flatness of the current $J(x)$, a natural object to consider is the parallel transporter on the world-sheet with $J(x)$ as connection, and in particular the *monodromy matrix* defined as the parallel transporter around a curve $c_{\sigma,\tau}$ bound at (σ,τ) and winding once around the world-sheet

$$\Omega(x,\sigma,\tau) = P\overleftarrow{\exp}\int_{[c_{\sigma,\tau}]} J(x),$$

which only depends on the homotopy class $[c_{\sigma,\tau}]$ of the curve $c_{\sigma,\tau}$ with both end-points fixed at (σ,τ). An immediate property of $\Omega(x,\sigma,\tau)$ is that its (σ,τ)-evolution is isospectral, i.e.

$$\Omega(x,\sigma',\tau') = U\Omega(x,\sigma,\tau)U^{-1}, \quad \text{where } U = P\overleftarrow{\exp}\int_{(\sigma,\tau)}^{(\sigma',\tau')} J(x). \tag{2.16}$$

This leads straight away to a way of generating infinitely many conserved charges from traces of powers of monodromy matrices since

$$\partial_\sigma \text{tr}\ \Omega(x)^n = \partial_\tau \text{tr}\ \Omega(x)^n = 0.$$

2.4 Involution of conserved charges

The statement of the involution property of the conserved charges generated by tr $\Omega(x)^n$ is equivalent to the statement

$$\left\{\text{tr}\ \Omega(x)^n, \text{tr}\ \Omega(x')^m\right\} = 0, \quad \forall n,m \in \mathbb{N}.$$

In order to show this we must first obtain the Poisson bracket algebra of monodromy matrices $\{\Omega(x) \overset{\otimes}{,} \Omega(x')\}$, which is the main focus of this subsection and appendix A. However, as we review below, since the original Poisson brackets (2.12b) of the current contain a non-ultralocal term, the resulting brackets of monodromy matrices are ambiguous and require regularisation.

2.4.1 Algebra of Lax connections

The space component of the Lax connection $J(x,\sigma,\tau)$ defined in section 2.3 is given by

$$J_1(\sigma,x) = \frac{1}{1-x^2}(j_1(\sigma) + x j_0(\sigma)).$$

The monodromy matrix being the path ordered exponential of the space component J_1, we will need the Poisson bracket $\{J_1, J_1\}$ in order to construct the Poisson bracket of monodromy matrices. So consider,

$$\begin{aligned}
\frac{\sqrt{\lambda}}{4\pi}\left\{J_1^a(\sigma,x), J_1^b(\sigma',x')\right\} &= -\frac{\sqrt{\lambda}}{4\pi}\frac{1}{(1-x^2)(1-x'^2)}\left\{j_1^a(\sigma) + x j_0^a(\sigma), j_1^b(\sigma') + x' j_0^b(\sigma')\right\} \\
&= -\frac{1}{(1-x^2)(1-x'^2)}\left[(x+x')\left(f^{abc} j_1^c(\sigma)\delta(\sigma-\sigma') + \delta^{ab}\delta'(\sigma-\sigma')\right) + xx' f^{abc} j_0^c(\sigma)\delta(\sigma-\sigma')\right] \\
&= -\frac{x+x'}{(1-x^2)(1-x'^2)}\delta^{ab}\delta'(\sigma-\sigma') - \frac{1}{x-x'}\left[\frac{x^2}{1-x^2}J_1^c(\sigma,x') - \frac{x'^2}{1-x'^2}J_1^c(\sigma,x)\right] f^{abc}\delta(\sigma-\sigma').
\end{aligned}$$

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Now we switch to tensor notation by contracting both sides with $t^a \otimes t^b$ and using $[\eta, t^c \otimes \mathbf{1}] = -f^{abc} t^a \otimes t^b$, $[\eta, \mathbf{1} \otimes t^c] = f^{abc} t^a \otimes t^b$, where $\eta := -t^a \otimes t^a$, so that

$$\frac{\sqrt{\lambda}}{4\pi} \left\{ J_1(\sigma, x) \stackrel{\otimes}{,} J_1(\sigma', x') \right\} = \left[-\frac{\eta}{x - x'}, \frac{x'^2}{1 - x'^2} J_1(\sigma, x) \otimes \mathbf{1} + \frac{x^2}{1 - x^2} \mathbf{1} \otimes J_1(\sigma, x') \right] \delta(\sigma - \sigma') + \frac{x + x'}{(1 - x^2)(1 - x'^2)} \eta \delta'(\sigma - \sigma').$$

This bracket has the general form of the fundamental Poisson bracket $\{J_1, J_1\}$ for a non-ultralocal integrable system formulated by Maillet [20, 21]

$$\begin{aligned} \left\{ J_1(\sigma, x) \stackrel{\otimes}{,} J_1(\sigma', x') \right\} &= r'(\sigma, x, x') \delta(\sigma - \sigma') \\ &\quad + \left[r(\sigma, x, x'), J_1(\sigma, x) \otimes \mathbf{1} + \mathbf{1} \otimes J_1(\sigma', x') \right] \delta(\sigma - \sigma') \\ &\quad - \left[s(\sigma, x, x'), J_1(\sigma, x) \otimes \mathbf{1} - \mathbf{1} \otimes J_1(\sigma', x') \right] \delta(\sigma - \sigma') \\ &\quad - \left(s(\sigma, x, x') + s(\sigma', x, x') \right) \delta'(\sigma - \sigma'), \end{aligned} \tag{2.17}$$

where in our case $s(x, x') = -\frac{2\pi}{\sqrt{\lambda}} \frac{x + x'}{(1 - x^2)(1 - x'^2)} \eta$ is constant (independent of σ and τ), and we find that r is constant as well and given by

$$r(x, x') = -\frac{2\pi}{\sqrt{\lambda}} \frac{x^2 + x'^2 - 2x^2 x'^2}{(x - x')(1 - x^2)(1 - x'^2)} \eta. \tag{2.18}$$

The principal chiral model was first described in terms of Maillet's $(r - s)$-matrix formalism in [19].

2.4.2 Algebra of monodromy matrices: Maillet regularisation

A first step towards obtaining the algebra of monodromy matrices is to consider first the algebra of transition matrices. A transition matrix is defined relative to an interval $[\sigma_1, \sigma_2]$ as follows

$$T(\sigma_1, \sigma_2, x) = P\overleftarrow{\exp} \int_{\sigma_2}^{\sigma_1} d\sigma J_1(\sigma, x).$$

The monodromy matrix is then simply a special transition matrix whose interval wraps the circle fully once, i.e.

$$\Omega(x, \sigma) = T(\sigma, \sigma + 2\pi, x).$$

Since the derivation of the algebra of transition matrices is fairly standard [2] we have left it to appendix A to avoid cluttering this section with algebra. The end result is the following bracket between two transition matrices with distinct intervals,

$$\begin{aligned} \left\{ T(\sigma_1, \sigma_2, x) \stackrel{\otimes}{,} T(\sigma'_1, \sigma'_2, x') \right\} &= +\epsilon(\sigma'_1 - \sigma'_2) \chi(\sigma; \sigma'_1, \sigma'_2) \times T(\sigma_1, \sigma, x) \otimes T(\sigma'_1, \sigma, x') \\ &\quad \times \left(r(\sigma, x, x') - s(\sigma, x, x') \right) T(\sigma, \sigma_2, x) \otimes T(\sigma, \sigma'_2, x') \Big|_{\sigma = \sigma_2}^{\sigma = \sigma_1} \\ &\quad + \epsilon(\sigma_1 - \sigma_2) \chi(\sigma; \sigma_1, \sigma_2) \times T(\sigma_1, \sigma, x) \otimes T(\sigma'_1, \sigma, x') \\ &\quad \times \left(r(\sigma, x, x') + s(\sigma, x, x') \right) T(\sigma, \sigma_2, x) \otimes T(\sigma, \sigma'_2, x') \Big|_{\sigma = \sigma'_2}^{\sigma = \sigma'_1}. \end{aligned}$$

It follows from this algebra that the function,

$$\Delta^{(1)}(\sigma_1, \sigma_2, \sigma_1', \sigma_2'; x, x') = \{T(\sigma_1, \sigma_2, x) \overset{\otimes}{,} T(\sigma_1', \sigma_2', x')\}$$

is well defined and continuous where $\sigma_1, \sigma_2, \sigma_1', \sigma_2'$ are all distinct, but it has discontinuities proportional to $2s$ precisely across the hyperplanes corresponding to some of the $\sigma_1, \sigma_2, \sigma_1', \sigma_2'$ being equal. Defining the Poisson bracket $\{T \overset{\otimes}{,} T\}$ for coinciding intervals ($\sigma_1 = \sigma_1', \sigma_2 = \sigma_2'$) or adjacent intervals ($\sigma_1' = \sigma_2$ or $\sigma_1 = \sigma_2'$) requires defining the value of the discontinuous matrix-valued function $\Delta^{(1)}$ at its discontinuities. It is shown in [20] that requiring antisymmetry of the Poisson bracket and the derivation rule to hold imposes the symmetric definition of $\Delta^{(1)}$ at its discontinuous points; for example at $\sigma_1 = \sigma_1'$ we must define

$$\Delta^{(1)}(\sigma_1, \sigma_2, \sigma_1, \sigma_2'; x, x') = \\ \lim_{\epsilon \to 0^+} \frac{1}{2} \left(\Delta^{(1)}(\sigma_1, \sigma_2, \sigma_1 + \epsilon, \sigma_2'; x, x') + \Delta^{(1)}(\sigma_1, \sigma_2, \sigma_1 - \epsilon, \sigma_2'; x, x') \right),$$

and likewise for all other possible coinciding endpoints. This definition is equivalent to assigning the value of $\frac{1}{2}$ to the characteristic function χ at its discontinuities. Having thus defined $\Delta^{(1)}$ at its discontinuities we now have a definition of the Poisson bracket $\{T \overset{\otimes}{,} T\}$ for coinciding and adjacent intervals consistent with the antisymmetry of the Poisson bracket and the derivation rule. However this definition of the $\{T \overset{\otimes}{,} T\}$ Poisson bracket does not satisfy the Jacobi identity as is shown in [20], so that in fact no strong definition of the bracket $\{T \overset{\otimes}{,} T\}$ with coinciding or adjacent intervals can be given without violating the Jacobi identity [20]. It is nevertheless possible [20, 18] to give a weak[4] definition of this bracket for coinciding or adjacent intervals in a way that is consistent with the Jacobi identity as follows: consider the multiple Poisson bracket of $(n+1)$ transition matrices

$$\Delta^{(n)} \left(\sigma_1^{(1)}, \sigma_2^{(1)}, \ldots, \sigma_1^{(n+1)}, \sigma_2^{(n+1)}; x^{(1)}, \ldots, x^{(n+1)} \right) = \\ \left\{ T \left(\sigma_1^{(1)}, \sigma_2^{(1)}, x^{(1)} \right) \overset{\otimes}{,} \left\{ \ldots \overset{\otimes}{,} \left\{ T \left(\sigma_1^{(n)}, \sigma_2^{(n)}, x^{(n)} \right) \overset{\otimes}{,} T \left(\sigma_1^{(n+1)}, \sigma_2^{(n+1)}, x^{(n+1)} \right) \right\} \ldots \right\} \right\},$$

which is unambiguously defined and continuous where $\sigma_1^{(1)}, \sigma_2^{(1)}, \ldots, \sigma_1^{(n+1)}, \sigma_2^{(n+1)}$ are all distinct, but again is discontinuous across the hyperplanes defined by some of the points $\sigma_1^{(1)}, \sigma_2^{(1)}, \ldots, \sigma_1^{(n+1)}, \sigma_2^{(n+1)}$ being equal. The values of $\Delta^{(n)}$ at its discontinuities are defined by employing a point splitting regularisation followed by a total symmetrisation limit [20]. For example, we define its value at $\sigma_1^{(i)} = \sigma_1, i = 1, \ldots, n+1$ by

$$\Delta^{(n)} \left(\sigma_1, \sigma_2^{(1)}, \ldots, \sigma_1, \sigma_2^{(n+1)}; x^{(1)}, \ldots, x^{(n+1)} \right) = \\ \lim_{\epsilon \to 0^+} \frac{1}{(n+1)!} \sum_{p \in S_{n+1}} \Delta^{(n)} \left(\sigma_1 + p(1)\epsilon, \sigma_2^{(1)}, \ldots, \sigma_1 + p(n+1)\epsilon, \sigma_2^{(n+1)}; x^{(1)}, \ldots, x^{(n+1)} \right),$$

[4]The bracket is weak in the sense that any multiple Poisson bracket of T's can be given a meaning which cannot be reduced to its similarly defined constituent Poisson brackets, i.e. the multiple Poisson bracket $\{T \overset{\otimes}{,} \{\ldots \{T \overset{\otimes}{,} T\} \ldots\}\}$ with n factors of T must be separately defined for each n.

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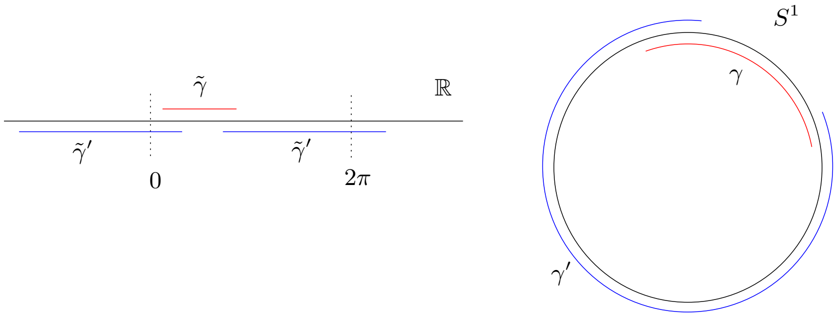

Figure 2: Example of a path lifting required in computing Poisson brackets of transition matrices on S^1 of the form $\{T(\gamma,x) \stackrel{\otimes}{,} T(\gamma',x')\}$.

and similarly one defines the value of $\Delta^{(n)}$ at all other discontinuities. With the function $\Delta^{(n)}$ being defined at its discontinuities we now have the definition of a weak bracket which reduces to the normal Poisson bracket on quantities for which the latter is continuous. It is shown in [20] that the Jacobi identity for transition matrices with coinciding or adjacent interval is now satisfied in terms of this weak bracket ($\Delta^{(2)}$ being the relevant quantity in this case).

Using this regularisation procedure we now derive an expression for the Poisson bracket between two monodromy matrices in the periodic case under consideration, a result which was first obtained in [18, 20]. To begin with consider the Poisson bracket $\{T(\gamma,x) \stackrel{\otimes}{,} T(\gamma',x')\}$ between two generic transition matrices $T(\gamma,x)$ and $T(\gamma',x')$ on the circle S^1, defined relative to two different paths γ and γ' on S^1, e.g.

$$T(\gamma,x) = P\overleftarrow{\exp} \int_{\gamma} d\sigma J_1(\sigma,x). \tag{2.19}$$

We would like to compute this bracket by working on the universal cover $\mathbb{R}$ of S^1. So we choose a lift $\tilde{\gamma}$ of the path γ to $\mathbb{R}$. Then because the only contribution to the Poisson bracket comes from the region of overlap between γ and γ' on S^1 (by (A.4) and (2.19)), we have that

$$\{T(\gamma,x) \stackrel{\otimes}{,} T(\gamma',x')\} = \sum_{\tilde{\gamma}' \text{ lift of } \gamma'} \{T(\tilde{\gamma},x) \stackrel{\otimes}{,} T(\tilde{\gamma}',x')\}, \tag{2.20}$$

where the sum is over lifts $\tilde{\gamma}'$ of γ' to $\mathbb{R}$. An example of these lifted paths is shown in figure 2.

Let us now apply this formula to compute the Poisson bracket between two transition matrices $\Omega(x,\sigma)$ and $\Omega(x',\sigma)$ on S^1. The common interval γ of both matrices stretches once around the full circle and so it follows that if we take $\tilde{\gamma} = [\sigma, \sigma + 2\pi]$ to be the lift of the interval of $\Omega(x,\sigma)$ then there are only three possibilities for the lift $\tilde{\gamma}'$ of the interval of $\Omega(x',\sigma)$ which give a non-zero contribution to the right hand side of (2.20), namely

$$[\sigma - 2\pi, \sigma], \quad [\sigma, \sigma + 2\pi], \quad [\sigma + 2\pi, \sigma + 4\pi]. \tag{2.21}$$

Since the corresponding three brackets $\{T(\tilde{\gamma},x) \stackrel{\otimes}{,} T(\tilde{\gamma}',x')\}$ on $\mathbb{R}$ are over coinciding or adjacent intervals they need to be regularised by the procedure described above. Let us start by considering the coinciding interval bracket $\{T(\sigma,\sigma{+}2\pi,x) \overset{\otimes}{,} T(\sigma,\sigma{+}2\pi,x')\}$. There are 4 different possible point splittings of the endpoints, each giving the same contribution (using (A.6))

$$r(x,x')\left(\Omega(x,\sigma)\otimes\Omega(x',\sigma)\right) - \left(\Omega(x,\sigma)\otimes\Omega(x',\sigma)\right)r(x,x')$$

in the limit of coinciding points. On the other hand, the adjacent interval brackets (corresponding to the first and last choices for $\tilde{\gamma}'$ in (2.21)) each have two possible point splittings and together they contribute, in the coinciding end-point limit,

$$(\Omega(x,\sigma)\otimes\mathbf{1})\,s(x,x')\left(\mathbf{1}\otimes\Omega(x',\sigma)\right) - \left(\mathbf{1}\otimes\Omega(x',\sigma)\right)s(x,x')\,(\Omega(x,\sigma)\otimes\mathbf{1})$$

to the Poisson bracket of two monodromy matrices. The sum of the last two expressions gives the right hand side of (2.20) which yields the sought-after (weak) Poisson bracket between two monodromy matrices on S^1

$$\begin{aligned}\left\{\Omega(x,\sigma) \overset{\otimes}{,} \Omega(x',\sigma)\right\} &= \left[r(x,x'),\Omega(x,\sigma)\otimes\Omega(x',\sigma)\right]\\ &\quad+ (\Omega(x,\sigma)\otimes\mathbf{1})\,s(x,x')\left(\mathbf{1}\otimes\Omega(x',\sigma)\right)\\ &\quad- \left(\mathbf{1}\otimes\Omega(x',\sigma)\right)s(x,x')\,(\Omega(x,\sigma)\otimes\mathbf{1})\,. \end{aligned} \tag{2.22}$$

Consider now the bracket $\{\Omega(x,\sigma)^n \overset{\otimes}{,} \Omega(x',\sigma)^m\}$ for any $n,m\in\mathbb{N}$, which can easily be reduced to the previous Poisson bracket as follows (omitting the σ-dependence)

$$\left\{\Omega(x)^n \overset{\otimes}{,} \Omega(x')^m\right\} = nm\left(\Omega(x)^{n-1}\otimes\mathbf{1}\right)\left\{\Omega(x)\overset{\otimes}{,}\Omega(x')\right\}\left(\mathbf{1}\otimes\Omega(x)^{m-1}\right).$$

Then using the standard notational shorthands $\overset{1}{A} = A\otimes\mathbf{1}$ and $\overset{2}{A} = \mathbf{1}\otimes A$, and taking the trace over both factors of the tensor product we find

$$\begin{aligned}\left\{\mathrm{tr}\;\Omega(x)^n, \mathrm{tr}\;\Omega(x')^m\right\} &= nm\;\mathrm{tr}_{12}\left(\overset{1}{\Omega}(x)^{n-1}\overset{2}{\Omega}(x')^{m-1}\left\{\overset{1}{\Omega}(x),\overset{2}{\Omega}(x')\right\}\right)\\ &= nm\;\mathrm{tr}_{12}\left[r(x,x')+s(x,x'),\overset{1}{\Omega}(x)^n\overset{2}{\Omega}(x')^m\right]\\ \text{i.e.}\;\; \left\{\mathrm{tr}\;\Omega(x)^n, \mathrm{tr}\;\Omega(x')^m\right\} &= 0,\end{aligned}$$

where in the second line we have used (2.22). Because this last bracket is zero it can be understood as defining a bracket in the strong sense and without recourse to any regularisation. We deduce from this last relation that the invariants of the system encoded in the quantity tr $\Omega(x)^n$ are in involution with respect to the Poisson bracket; this is the full statement of Liouville integrability of the system.

As a specific check of (2.22) we show that the $\mathrm{SU}(2)_R$ symmetry is canonically realised on $\Omega(x)$ via the weak Poisson bracket [18]. It is straightforward to show that the monodromy matrix has the following asymptotics at $x=\infty$ [1],

$$\Omega(x,\sigma,\tau) = \mathbf{1} + \frac{1}{x}\frac{4\pi Q_R}{\sqrt{\lambda}} + O\left(\frac{1}{x^2}\right) \quad \text{as } x\to\infty.$$

Then starting with equation (2.22) multiplied by $x\,(\epsilon \otimes \mathbf{1})$ and taking the trace over the first tensor product space followed by the limits $x \to \infty$ and $x' \to 0$ one deduces, using also the asymptotics $r(x,x') \sim_{x\to\infty} \frac{2\pi}{\sqrt{\lambda}} \frac{1-2x'^2}{x(1-x'^2)}$ and $s(x,x') \sim_{x\to\infty} \frac{2\pi}{\sqrt{\lambda}} \frac{1}{x(1-x'^2)}$, that

$$\left\{\epsilon \cdot Q_R, \Omega(x')\right\} = \left[\epsilon, \Omega(x')\right].$$

In other words, the right Noether charge Q_R generates the correct transformation on $\Omega(x)$, which we expect to be

$$\Omega(x) \to U_R^{-1} \Omega(x) U_R,$$

provided we use the weak bracket instead of the Poisson bracket.

3. Symplectic structure for finite-gap solutions

In a previous paper [1] we constructed the general finite-gap solution to the equations of motion of a string moving on $\mathbb{R} \times S^3$ satisfying the Virasoro constraints. We also constructed the corresponding moduli space of solutions. Our aim here is to determine the symplectic structure induced on the moduli space of solutions by the regularised Poisson brackets obtained in the previous section. As we will see below, our analysis for strings moving on $\mathbb{R} \times S^3$ can be thought of as a non-linear generalisation of the more familiar Hamiltonian analysis of strings in flat space. We will therefore begin by reviewing the standard discussion of the flat space case following eg section 2 of [10].

We will consider a closed bosonic string moving on $(D+1)$-dimensional Minkowski space with worldsheet fields $X^\mu(\sigma,\tau)$ for $\mu = 0,1,\ldots,D$. In conformal gauge, the world-sheet equation of motion is the two-dimensional Laplace equation $\partial_+\partial_- X^\mu = 0$. As the equation is linear, the general solution for closed string boundary conditions is given by the Fourier series,

$$X^\mu(\sigma,\tau) = x^\mu + p^\mu \tau + i \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in(\tau-\sigma)} + i \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)}. \tag{3.1}$$

where the Fourier coefficients α_n^μ and $\tilde{\alpha}_n^\mu$ correspond to classical oscillator coordinates for left- and right-moving modes respectively. For our purposes it will be convenient to restrict our attention to classical solutions with a finite number of oscillators turned on. Generic solutions can then be obtained as a limiting case. We will see that these 'finite-oscillator' solutions are close analogs of the finite-gap solutions of string theory on $\mathbb{R} \times S^3$ and other classically integrable backgrounds.

Since (3.1) is the general solution to the field equations, the fields $X^\mu(\sigma) = X^\mu(\sigma,0)$ and $P^\mu(\sigma) = \dot{X}^\mu(\sigma,0)$ restricted to a τ-slice (taken at $\tau = 0$ without loss of generality) give a convenient parametrisation of the phase space of the string.[5] In terms of the oscillator coordinates we find,

$$\begin{aligned} X^\mu(\sigma) &= x^\mu + i \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{in\sigma} + i \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-in\sigma}, \\ P^\mu(\sigma) &= p^\mu + \sum_{n\neq 0} \alpha_n^\mu e^{in\sigma} + \sum_{n\neq 0} \tilde{\alpha}_n^\mu e^{-in\sigma}. \end{aligned} \tag{3.2}$$

[5] Note that this is not the physical phase space as we have not yet imposed the Virasoro constraints.

Conversely the oscillator coefficients $\alpha^\mu_n, \tilde{\alpha}^\mu_n$ as well as the centre of mass position and momenta x^μ, p^μ can be extracted from a generic phase-space configuration $X^\mu(\sigma), P^\mu(\sigma)$ by the following relations

$$\begin{cases} \alpha^\mu_m = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{im\sigma}\dfrac{1}{2}\left(P^\mu(\sigma) - \partial_\sigma X^\mu(\sigma)\right)d\sigma, \quad m \neq 0 \\ \tilde{\alpha}^\mu_m = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{im\sigma}\dfrac{1}{2}\left(P^\mu(\sigma) + \partial_\sigma X^\mu(\sigma)\right)d\sigma, \quad m \neq 0 \\ x^\mu = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} X^\mu(\sigma)d\sigma, \qquad p^\mu = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} P^\mu(\sigma)d\sigma. \end{cases} \tag{3.3}$$

Equations (3.3) are the inverse of the equations (3.2) and the transformation

$$\{X^\mu(\sigma), P^\mu(\sigma)\} \mapsto \{x^\mu, p^\mu, \alpha^\mu_n, \tilde{\alpha}^\mu_n\} \tag{3.4}$$

is simply a change of variable on phase-space. The Poisson brackets which follow from the string action, take the form,

$$\{X^\mu(\sigma), X^\nu(\sigma')\} = \{P^\mu(\sigma), P^\nu(\sigma')\} = 0, \tag{3.5}$$

$$\{P^\mu(\sigma), X^\nu(\sigma')\} = \eta^{\mu\nu}\delta(\sigma - \sigma'), \tag{3.6}$$

and it is straightforward to rewrite these brackets in the new coordinate system as,

$$\begin{aligned} \{\alpha^\mu_m, \alpha^\nu_n\} &= im\delta_{m+n}\eta^{\mu\nu}, \{\alpha^\mu_m, \tilde{\alpha}^\nu_n\} = 0, \\ \{\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n\} &= im\delta_{m+n}\eta^{\mu\nu}, \{p^\mu, x^\nu\} = \eta^{\mu\nu}. \end{aligned} \tag{3.7}$$

So far we have discussed the full solution space of the equations of motion. The next step is to restrict to physical configurations of the string by imposing the Virasoro constraints and fixing the residual gauge symmetry. These steps are easily accomplished by imposing light-cone gauge. We begin by defining light-cone coordinate $X^\pm = X^0 \pm X^D$ and imposing the gauge condition $X^+ = p^+\tau + x^+$. With this choice, it is possible to solve the Virasoro constraint explicitly to eliminate the longitudinal oscillator coordinates. The remaining independent degrees of freedom are,

$$\{x^i, p^i, x^-, p^-, \alpha^i_n, \tilde{\alpha}^i_n\} \tag{3.8}$$

where the index $i = 1, 2, \ldots, D-1$ runs over the transverse spacetime dimensions. To find the Poisson brackets of the physical degrees of freedom one must follow the standard Dirac procedure for constrained systems. In the present case this is described in detail in the book by Brink and Henneaux [11]. The Virasoro constraint and the light cone gauge fixing condition together correspond to a system of second order constraints on the phase space. Fortunately, the resulting Dirac bracket for the transverse degrees of freedom is the same as their naive Poisson bracket,

$$\begin{aligned} \{\alpha^i_m, \alpha^j_n\} &= im\delta_{m+n}\delta^{ij}, \{\alpha^i_m, \tilde{\alpha}^j_n\} = 0, \\ \{\tilde{\alpha}^i_m, \tilde{\alpha}^j_n\} &= im\delta_{m+n}\delta^{ij}, \{p^i, x^j\} = \delta^{ij}. \end{aligned} \tag{3.9}$$

These Poisson brackets are the starting point for canonical quantisation of the string which proceeds by the usual recipe of promoting Poisson brackets to commutators.

Classical string theory in flat space is trivially integrable as the corresponding equation of motion is linear. For comparison with the non-linear case, it will be convenient to exhibit integrability explicitly by constructing the corresponding action-angle variables. As the dynamics of the COM degrees of freedom of the string is free we will focus on the transverse oscillators which describe the physical excitations of the string in its rest frame. We introduce a new set of variables $\{\theta_n^j, S_n^j, \tilde{\theta}_n^j, \tilde{S}_n^j\}$ for $j = 1, \ldots, D-1$ via the relations,

$$\alpha_n^j = \sqrt{n S_n^j}\, e^{i\theta_n^j}, \tilde{\alpha}_n^j = \sqrt{n \tilde{S}_n^j}\, e^{i\tilde{\theta}_n^j}.$$

The variables S_n^j and $\tilde{S}_n^j$ are classical analogs of the occupation numbers of each transverse oscillator. These variables are trivially time independent and therefore correspond to conserved charges. One may also check the involution condition,

$$\{S_n^i, S_m^j\} = \{S_n^i, \tilde{S}_m^j\} = \{\tilde{S}_n^i, \tilde{S}_m^j\} = 0 \tag{3.10}$$

These are the action variables of the flat space string.

The angular variables θ_n^j and $\tilde{\theta}_n^j$ each have period 2π and are canonically conjugate to the corresponding action variables. Their non-vanishing Poisson brackets are,

$$\{S_n^i, \theta_m^j\} = \{\tilde{S}_n^i, \tilde{\theta}_m^j\} = \delta_{nm}\delta^{ij} \tag{3.11}$$

It follows immediately from Hamilton's equations that the angle variables evolve linearly in time while, as above, the conjugate action variables remain constant.

$$\begin{aligned} \theta_n^\mu(\tau) &= \theta_n^\mu(0) - n\tau, \quad & S_n^\mu(\tau) &= S_n^\mu(0) = \text{const.} \\ \tilde{\theta}_n^\mu(\tau) &= \tilde{\theta}_n^\mu(0) - n\tau, \quad & \tilde{S}_n^\mu(\tau) &= \tilde{S}_n^\mu(0) = \text{const.} \end{aligned}$$

We now turn to the case at hand of string on $\mathbb{R} \times S^3$ and present a non-linear analogue of the above concepts for strings on flat space. However we will proceed in a slightly different order. As we have already fixed the gauge completely in section 2.1 by imposing static gauge $X_0 = \kappa\tau$, we now solve both the equations of motion and the Virasoro constraint simultaneously through algebro-geometric methods in section 3.1. In this way we immediately obtain the general 'finite-gap' solution (analogue of the 'finite-oscillator' solution above) expressed directly in terms of physical degrees of freedom. Section 3.2 aims to derive the analogues of (3.3) in the case of the nonlinear differential equations for a string on $\mathbb{R} \times S^3$ which will be crucial in section 3.3 for determining Poisson brackets on the algebro-geometric data. Finally, in section 3.4 we define the change of variable to action-angle coordinates and verify the canonical Poisson brackets for these variables.

3.1 Finite-gap integration

In this subsection we will briefly review the explicit construction of finite-gap solutions given in [1]. The reader should consult this reference for extra details.

3.1.1 The spectral curve

The starting point for the method of finite-gap integration is the formulation of the equations of motion (2.3) of the system as the flatness condition (2.15). Representing the equations of motion in this form introduces a large amount of spurious symmetries which we are free to fix as we proceed; indeed, equation (2.15) is invariant under gauge transformations $J(x) \mapsto \tilde{g}J(x)\tilde{g}^{-1} + d\tilde{g}\tilde{g}^{-1}$.

Now the isospectral (σ,τ)-evolution (2.16) of the monodromy matrix leads naturally to the definition of a (σ,τ)-independent *spectral curve* in $\mathbb{C}^2$

$$\Gamma: \quad \Gamma(x,y) = \det\left(y\mathbf{1} - \Omega(x,\sigma,\tau)\right) = 0.$$

However, this curve is highly singular [1] and so one should replace it with an algebraic curve Σ defined as a desingularisation of Γ (for details of this construction see [1, 7]). An important property of the spectral curve Γ is that above any non-singular point $(x,y) \in \Gamma$ ($d\Gamma(x,y) \neq 0$) there is a unique eigenvector of $\Omega(x)$ with eigenvalue y. It follows that the desingularised curve Σ has a unique eigenvector above any of its points. In the present case where $\Omega(x)$ is 2×2 the curve Σ is also hyperelliptic with projection denoted $\hat{\pi}: \Sigma \to \mathbb{CP}^1$; we also introduce the notation $\{x^{\pm}\} = \hat{\pi}^{-1}(x)$ for the set of points above $x \in \mathbb{CP}^1$.

The curve Σ is naturally equipped with a normalised second kind Abelian differential dp, with singularities only at the points $\{(+1)^{\pm}, (-1)^{\pm}\}$ above $x = \pm 1$, specified uniquely by

$$\int_{a_i} dp = 0, \qquad \int_{b_i} dp = 2\pi k_i \in \mathbb{Z},$$

$$\begin{aligned} dp(x^{\pm}) &= \mp d\left(\frac{\pi\kappa_+}{x-1}\right) + O\left((x-1)^0\right) \quad \text{as } x \to +1, \\ dp(x^{\pm}) &= \mp d\left(\frac{\pi\kappa_-}{x+1}\right) + O\left((x+1)^0\right) \quad \text{as } x \to -1, \end{aligned} \tag{3.12}$$

where $\{a_i, b_i\}_{i=1}^g$ is a canonical basis of $H_1(\Sigma,\mathbb{Z})$. The asymptotics of dp near the points $\{0^{\pm}, \infty^{\pm}\}$ can be deduced [1] from the 'highest weight' conditions (2.11) and are directly related to the Casimirs R^2, L^2 of $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L$

$$\begin{aligned} dp(x^{\pm}) &= \mp d\left[\frac{1}{x}\frac{2\pi R}{\sqrt{\lambda}} + O\left(\frac{1}{x^2}\right)\right], \quad \text{as } x \to \infty, \\ dp(x^{\pm}) &= \pm d\left[x\frac{2\pi L}{\sqrt{\lambda}} + O\left(x^2\right)\right], \quad \text{as } x \to 0. \end{aligned} \tag{3.13}$$

The Abelian integral $p(P) = \int_{\infty^+}^{P} dp$ is called the quasi-momentum and has the property that $\{e^{ip(x^+)}, e^{ip(x^-)}\}$ are the eigenvalues of $\Omega(x)$.

A convenient way of describing the moduli space of genus g curves Σ with punctures at $\{1^{\pm}, (-1)^{\pm}, 0^{\pm}, \infty^{\pm}\}$ and equipped with a meromorphic differential dp with specified behaviours (3.12), (3.13) near these punctures is as a leaf $\mathcal{L}$ in a foliation of the universal configuration space of [14]. To make contact with the construction of the universal configuration space of [14] we also introduce another meromorphic differential dz by specifying its Abelian integral

$$z = x + \frac{1}{x},$$

which is a single-valued function on Σ so that all periods of dz are zero (i.e. $\int_C dz = 0$ for any cycle $C \in H_1(\Sigma, \mathbb{R})$). The asymptotics of dz near the punctures are obvious from its definition. Full details of this construction can be found in [1]. However, in [1] we chose to keep R fixed, thereby describing only the internal degrees of freedom of the string by a leaf $\mathcal{L}|_R$ in a smooth g-dimensional foliation of the universal moduli space; in the present paper we allow R to vary so the leaf $\mathcal{L}$ under consideration will now have one extra dimension.

Using the set of local coordinates on this universal configuration space introduced in [14] the leaf in question is obtained as the joint level set of all but $g+1$ of the coordinates. Defining the following differential on Σ

$$\alpha = \frac{\sqrt{\lambda}}{4\pi} z dp, \tag{3.14}$$

the remaining $g+1$ coordinates parametrising the leaf are [14]

$$S_i = \frac{1}{2\pi i} \int_{a_i} \alpha, \ i = 1, \dots, g, \quad \frac{R}{2} = -\text{Res}_{\infty^+} \alpha. \tag{3.15}$$

Equivalently one can parametrise the moduli space $\mathcal{L}$ by assigning a *filling fractions* to each of the $K = g+1$ cuts $\mathcal{C}_I$

$$\mathcal{S}_I = \frac{1}{2\pi i} \int_{\mathcal{A}_I} \alpha, \ I = 1, \dots, K = g+1, \tag{3.16}$$

where $\mathcal{A}_I$ is a cycle encircling the cut $\mathcal{C}_I$ on the physical sheet. The filling fractions are related to the variable R and the parameter $\frac{L}{2} = \text{Res}_{0^+} \alpha$ by

$$\sum_{I=1}^{K} \mathcal{S}_I = \text{Res}_{\infty^+} \alpha + \text{Res}_{0^+} \alpha = \frac{1}{2}(L - R).$$

The moduli space $\mathcal{L}$ is therefore a complex manifold with only orbifold singularities of dimension

$$\dim \mathcal{L} = g + 1,$$

every point of which corresponds to an admissible pair (Σ, dp) where Σ has genus g.

3.1.2 The normalised eigenvector

Let us denote by $\boldsymbol{h}(P, \sigma, \tau)$ the unique normalised eigenvector of $\Omega(x, \sigma, \tau)$ at a point $P \in \Sigma$ above $x = \hat{\pi}(P)$, normalised by

$$\boldsymbol{\alpha} \cdot \boldsymbol{h}(P) = 1, \tag{3.17}$$

where we choose here[6] $\boldsymbol{\alpha} = (1, 1)$ following [3]. Its components are meromorphic functions on Σ and it follows from a standard argument that $\boldsymbol{h}(P, \sigma, \tau)$ has $g+1$ poles[7] on Σ in

[6] In contrast to [1] where the normalisation $\boldsymbol{\alpha} = (1, 0)$ was used. Changing the normalisation of $\boldsymbol{h}$ will obviously have no effect on the reconstructed solution since this is constructed out of a vector proportional to $\boldsymbol{h}$ anyway.

[7] A vector $\boldsymbol{v}(P)$ on Σ is said to have a pole at $Q \in \Sigma$ if at least one of its components has a pole at Q.

the present case, which we denote by $\hat{\gamma}(\sigma,\tau)$. At this point we can fix some of the gauge redundancy of (2.15) by using the gauge transformation $\boldsymbol{h} \mapsto H(\infty)^{-1}\boldsymbol{h}$ where $H(x) = (\boldsymbol{h}(x^+), \boldsymbol{h}(x^-))$ to set $\boldsymbol{h}(\infty^+) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \boldsymbol{h}(\infty^-) = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$; note that this gauge transformation preserves the normalisation of $\boldsymbol{h}$ because $\boldsymbol{\alpha} H(x) = \boldsymbol{\alpha}$. The residual gauge symmetry consists of diagonal matrices $\tilde{g}(\sigma,\tau) = \mathrm{diag}(d_1, d_2)$

$$\Omega(x) \mapsto \tilde{g}\Omega(x)\tilde{g}^{-1}, \quad \boldsymbol{h} \mapsto f(P)^{-1}\tilde{g}\boldsymbol{h}, \quad \text{where } f(P) = \boldsymbol{\alpha}\cdot(\tilde{g}\boldsymbol{h}(P)). \tag{3.18}$$

The role of the function $f(P)$ is to keep $\boldsymbol{h}$ normalised. It has the effect of changing the divisor $\hat{\gamma}(\sigma,\tau)$ of poles of $\boldsymbol{h}$ to the equivalent divisor $\hat{\gamma}'(\sigma,\tau)$ ($\sim \hat{\gamma}(\sigma,\tau)$) of zeroes of f. Given a divisor $\hat{\gamma}(\sigma,\tau)$ of degree $g+1$, the following analytic properties uniquely specify the components of $\boldsymbol{h}$ by the Riemann-Roch theorem

$$(h_1) \geq \hat{\gamma}(\sigma,\tau)^{-1}\infty^-, \quad h_1(\infty^+) = 1, \quad \text{and} \quad (h_2) \geq \hat{\gamma}(\sigma,\tau)^{-1}\infty^+, \quad h_2(\infty^-) = 1.$$

Note that the divisor of zeroes of h_1 is $\gamma(\sigma,\tau)\infty^-$ where $\gamma(\sigma,\tau)$ denotes the 'dynamical divisor' of degree g that was introduced in [1] (where $\boldsymbol{h}$ was normalised by $\frac{1}{h_1}$ forcing its second component to have poles at $\gamma(\sigma,\tau)\infty^-$).

The gauge fixing condition $\boldsymbol{h}(\infty^+) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \boldsymbol{h}(\infty^-) = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ imposed so far also fixes part of the global $\mathrm{SU}(2)_R$ symmetry of the equations of motion by restricting the $\mathrm{SU}(2)_R$ current j to the level set $Q_R = \frac{1}{2i}R\sigma_3$. Indeed, the constant part of (3.18) corresponds to the unfixed $\mathrm{U}(1)_R$ subgroup of the global $\mathrm{SU}(2)_R$ symmetry group (in fact, before having imposed reality conditions we are really dealing with a $\mathbb{C}^*$ subgroup of $\mathrm{SL}(2,\mathbb{C})_R$). Let us end this section by showing that the choice of an initial value for the $\mathrm{U}(1)_R$ angle corresponds exactly to the choice of a representative of the equivalence class $[\hat{\gamma}(0,0)]$.

Since a specific representative $\hat{\gamma}'(0,0) = \prod_{i=1}^{g+1}\hat{\gamma}'_i \sim \hat{\gamma}(0,0)$ of the equivalence class $[\hat{\gamma}(0,0)]$ is uniquely specified by a single one of its points it suffices to show that for any arbitrary point $\hat{\gamma}'_1 \in \Sigma$ there exists a unique diagonal $\tilde{g} \in \mathrm{SL}(2,\mathbb{C})_R$ such that $f(\hat{\gamma}'_1, 0, 0) = 0$. But a generic diagonal matrix $\tilde{g} \in \mathrm{SL}(2,\mathbb{C})_R$ takes the form

$$\tilde{g} = \begin{pmatrix} W & 0 \\ 0 & W^{-1} \end{pmatrix}, \tag{3.19}$$

and so the requirement that $f(\hat{\gamma}'_1, 0, 0) = 0$ simply reads $W\,h_1(\hat{\gamma}'_1, 0, 0) + W^{-1}\,h_2(\hat{\gamma}'_1, 0, 0) = 0$. The solution to this equation $W^2 = -h_2(\hat{\gamma}'_1, 0, 0)/h_1(\hat{\gamma}'_1, 0, 0)$ is unique (up to a trivial sign) and it follows that $\tilde{g}$ can be constructed uniquely in such a way that $f(\hat{\gamma}'_1, 0, 0) = 0$.

For later use we also identify reality conditions on the representative of the equivalence class $[\hat{\gamma}(0,0)]$, or more precisely on changes between representatives of $[\hat{\gamma}(0,0)]$. Given two equivalent divisors $\hat{\gamma}(0,0)$ and $\hat{\gamma}'(0,0) \sim \hat{\gamma}(0,0)$ which are the poles and zeroes of $f(P,0,0)$ respectively, the reality requirement $\tilde{g} \in \mathrm{SU}(2)_R$ imposes a restriction on the function $f(P,0,0)$ and hence on the allowed change of divisor $\hat{\gamma}(0,0) \to \hat{\gamma}'(0,0)$, namely $|W|^2 = 1$.

3.1.3 Vector Baker-Akhiezer functions

We now look for the analytic properties which uniquely specify the vector $\boldsymbol{\psi}$ solution to the consistency condition $(d - J(x))\,\boldsymbol{\psi} = 0$ of (2.15); once the solution to this system is known, the Lax connection can be recovered (when x does not correspond to a branch point) by $J(x) = d\hat{\Psi}(x) \cdot \hat{\Psi}(x)^{-1}$ where $\hat{\Psi}(x) = (\boldsymbol{\psi}(x^+), \boldsymbol{\psi}(x^-))$. Since the operators $d - J(x)$ and $\Omega(x)$ commute we can write $\boldsymbol{\psi}$ as

$$\boldsymbol{\psi}(P, \sigma, \tau) = \widehat{\Psi}(x, \sigma, \tau) \boldsymbol{h}(P, 0, 0), \tag{3.20}$$

where $\widehat{\Psi}(x, \sigma, \tau)$ is a formal matrix solution to $(d - J(x)) \widehat{\Psi}(x) = 0$. For definiteness, fix the initial condition to be $\boldsymbol{\psi}(P, 0, 0) = \boldsymbol{h}(P, 0, 0)$ so that $\widehat{\Psi}(x, 0, 0) = \mathbf{1}$ and hence $\widehat{\Psi}(x)$ satisfies $\widehat{\Psi}(x, \sigma, \tau)\Omega(x, 0, 0) = \Omega(x, \sigma, \tau)\widehat{\Psi}(x, \sigma, \tau)$. Because $J(x)$ has poles only at $x = \pm 1$, Poincaré's theorem on holomorphic differential equations implies that $\widehat{\Psi}(x, \sigma, \tau)$ is holomorphic outside $x = \pm 1$. Its singularities at $x = \pm 1$ are essential singularities of the form [1]

$$\widehat{\Psi}(x, \sigma, \tau) e^{-\widehat{S}_\pm(x, \sigma, \tau)} = O(1) \quad \text{in a neighbourhood of } x = \pm 1,$$

where the singular parts $\widehat{S}_\pm(x, \sigma, \tau) = \frac{i\kappa_\pm}{2} \frac{\tau \pm \sigma}{1 \mp x} \sigma_3 =: s_\pm(x, \sigma, \tau) \sigma_3$ were determined using the Virasoro constraints. Moreover, using the condition $J(\infty) = 0$ we see that $d\widehat{\Psi}(\infty, \sigma, \tau) = 0$ which implies $\widehat{\Psi}(\infty, \sigma, \tau) = \mathbf{1}$. This is enough to read off the analytic properties of $\boldsymbol{\psi}$ from its representation in the form (3.20) which uniquely specify its components as Baker-Akhiezer functions, namely

$$(\psi_1) \geq \hat{\gamma}(0,0)^{-1} \infty^-, \quad \psi_1(\infty^+) = 1, \quad \text{and} \quad (\psi_2) \geq \hat{\gamma}(0,0)^{-1} \infty^+, \quad \psi_2(\infty^-) = 1,$$
$$\text{with} \quad \begin{cases} \psi_i(x^\pm, \sigma, \tau) e^{\mp s_+(x, \sigma, \tau)} = O(1), & \text{as } x \to 1, \\ \psi_i(x^\pm, \sigma, \tau) e^{\mp s_-(x, \sigma, \tau)} = O(1), & \text{as } x \to -1. \end{cases}$$

Given this data which uniquely specifies the Baker-Akhiezer vector $\boldsymbol{\psi}$, one can reconstruct the Lax connection $J(x) = d\hat{\Psi}(x) \cdot \hat{\Psi}(x)^{-1}$ uniquely up to a residual gauge transformation. Changing $\hat{\gamma}(0,0)$ to an equivalent divisor $\hat{\gamma}'(0,0)$ amounts simply to a scaling $\boldsymbol{\psi} \to k\boldsymbol{\psi}$ by a function $k(P)$ with divisor $(k) = \hat{\gamma}(0,0) \cdot \hat{\gamma}'(0,0)^{-1}$, which has no effect on the reconstructed Lax connection $J(x) = d\hat{\Psi}(x) \cdot \hat{\Psi}(x)^{-1}$; therefore the equivalence class $[J(x)]$ of $J(x)$ under residual gauge transformations is uniquely specified by the equivalence class $[\hat{\gamma}(0,0)]$ and we have an injective map

$$[J(x)] \mapsto \{\Sigma, dp, [\hat{\gamma}(0,0)]\}. \tag{3.21}$$

Since the gauge fixing condition $J(\infty) = 0$ imposed above still allows for residual gauge transformations by constant diagonal matrices, corresponding precisely to the unfixed $\mathrm{U}(1)_R$ subgroup of the physical symmetry $\mathrm{SU}(2)_R$, the initial data pertaining to the $\mathrm{U}(1)_R$ symmetry cannot be determined by analytical considerations of the auxiliary linear system $(d - J(x))\boldsymbol{\psi} = 0$. The best we can do is simplify the injective map (3.21) down to the following injective map

$$[j] \mapsto \{\Sigma, dp, [\hat{\gamma}(0,0)]\}, \tag{3.22}$$

where $[j]$ denotes the equivalence class of j under $\mathrm{U}(1)_R$ conjugation. However, the $\mathrm{U}(1)_R$ initial angle was argued in section 3.1.2 to be fully specified by a choice of representative of the equivalence class $[\hat{\gamma}(0,0)]$. Thus the full set of initial data of a finite-gap solution can be completely specified by a divisor $\hat{\gamma}(0,0)$ of degree $\deg \hat{\gamma}(0,0) = g+1$. In other words we end up with the following injective map

$$j \mapsto \{\Sigma, dp, \hat{\gamma}(0,0)\}. \tag{3.23}$$

This is the analogue of the flat space equation (3.4) in the case at hand of the nonlinear equations of motion for a string moving on $\mathbb{R} \times S^3$.

This complete set of algebro-geometric data $\{\Sigma, dp, \hat{\gamma}(0,0)\}$ for an arbitrary finite-gap solution j can be succinctly described as a point in the bundle $\mathcal{M}_{\mathbb{C}}^{(2g+2)}$ over $\mathcal{L}$

$$S^{g+1}(\Sigma) \to \mathcal{M}_{\mathbb{C}}^{(2g+2)} \to \mathcal{L}, \tag{3.24}$$

whose fibre over every point of the base, specified by a curve Σ, is the $(g+1)$-st symmetric product $S^{g+1}(\Sigma) = \Sigma^{g+1}/S_{g+1}$ of Σ. If R where to be held fixed and the global $\mathrm{U}(1)_R$ symmetry factored out (as was the case in [1]), then the leaf would be reduced to $\mathcal{L}|_R$ (see section 3.1.1) and the $\mathrm{U}(1)_R$-reduced solution $[j]$ uniquely specified by the equivalence class $[\hat{\gamma}(0,0)]$ (see (3.22)) so that the relevant bundle in the $\mathrm{U}(1)_R$-reduced case is [1]

$$J(\Sigma) \to \mathcal{M}_{\mathbb{C}}^{(2g)} \to \mathcal{L}|_R,$$

using the Abel map $\boldsymbol{\mathcal{A}} : S^g(\Sigma) \to J(\Sigma)$ to identify each fibre with the Jacobian $J(\Sigma)$.

3.1.4 General finite-gap solution

Since the map (3.22) is injective (essentially by the Riemann-Roch theorem), it admits a left inverse

$$\{\Sigma, dp, [\hat{\gamma}(0,0)]\} \mapsto [j], \tag{3.25}$$

which takes a given set of admissible algebro-geometric data into a solution of the equations of motion (2.3) and the Virasoro constraints (2.8). This solution can be formally read off from the Lax connection $J(x) = (j - x * j)/(1-x^2)$ constructed out of the Baker-Akhiezer vector $\boldsymbol{\psi}$, namely

$$J(x) = d\hat{\Psi}(x) \cdot \hat{\Psi}(x)^{-1} \quad \text{with } \hat{\Psi}(x) = (\boldsymbol{\psi}(x^+), \boldsymbol{\psi}(x^-)), \tag{3.26}$$

as already mentioned in the previous section. However, the algebro-geometric reconstruction of the solution gives more than just a formal or implicit expression since vector Baker-Akhiezer functions on a Riemann surface Σ admit explicit representations in terms of Riemann θ-functions associated with Σ, thus enabling us to write down explicit formulae for the current j.

The analogue of the general flat space solution (3.1) with finitely many oscillators turned on, called a *finite-gap* solution, which solves both (2.3) and (2.8) was constructed in [1]. Its construction is based on an algebraic curve Σ of finite genus g and is given by the following expression for the light-cone components of the current j

$$\begin{aligned} j_{\pm}(\sigma,\tau) &= e^{\left(\frac{i}{2}\bar{\theta}_0 - \frac{i}{2}\int_{\infty^-}^{\infty^+} d\mathcal{Q}\right)\sigma_3} \Theta_{\pm}\left(\boldsymbol{\mathcal{A}}(\hat{\gamma}(0,0)) - \int_{\boldsymbol{b}} d\mathcal{Q}; \boldsymbol{\mathcal{A}}(\hat{\gamma}(0,0))\right) \\ &\quad \times (i\kappa_{\pm}\sigma_3)\, \Theta_{\pm}\left(\boldsymbol{\mathcal{A}}(\hat{\gamma}(0,0)) - \int_{\boldsymbol{b}} d\mathcal{Q}; \boldsymbol{\mathcal{A}}(\hat{\gamma}(0,0))\right)^{-1} e^{-\left(\frac{i}{2}\bar{\theta}_0 - \frac{i}{2}\int_{\infty^-}^{\infty^+} d\mathcal{Q}\right)\sigma_3}, \end{aligned} \tag{3.27}$$

where the notation used is defined as follows:

- The differential $d\mathcal{Q}(\sigma,\tau)$ is the unique normalised second kind Abelian differential with double poles at the points above $x = \pm 1$ of the prescribed form

$$d\mathcal{Q} \underset{x\to\pm1}{\sim} idS_\pm,$$

where $S_\pm(P,\sigma,\tau)$ are the singular parts of the problem defined as

$$\begin{cases} S_+(x^\pm,\sigma,\tau) = \mp\dfrac{i\kappa_+}{2}\dfrac{\sigma+\tau}{x-1}, \\ S_-(x^\pm,\sigma,\tau) = \mp\dfrac{i\kappa_-}{2}\dfrac{\sigma-\tau}{x+1}. \end{cases}$$

Note, the matrix $\widehat{S}_\pm(x,\sigma,\tau)$ defined in the pervious section is simply the diagonal matrix $\mathrm{diag}\left(-S_\pm(x^+,\sigma,\tau), -S_\pm(x^-,\sigma,\tau)\right)$.

- The divisor $\hat{\gamma}(0,0)$ is the divisor of poles of $\boldsymbol{h}(P,0,0)$ described in the previous sections. Its degree is $\deg\hat{\gamma}(0,0) = g+1$ and so it lives in the $(g+1)$-st symmetric product $S^{g+1}(\Sigma) = \Sigma^{g+1}/S_{g+1}$ of the curve Σ which is mapped surjectively onto the Jacobian $J(\Sigma)$ of Σ by means of the Abel map

$$\begin{aligned} \boldsymbol{\mathcal{A}} : S^{g+1}(\Sigma) &\to J(\Sigma) \\ \prod_{i=1}^{g+1} P_i &\mapsto 2\pi\sum_{i=1}^{g+1}\int_{\infty^+}^{P_i}\boldsymbol{\omega}. \end{aligned} \tag{3.28}$$

- The solution can only be recovered up to conjugation by constant diagonal matrices corresponding precisely to the $\mathbb{C}^*$ subgroup of $\mathrm{SL}(2,\mathbb{C})_R$ (which becomes the $\mathrm{U}(1)_R$ subgroup of $\mathrm{SU}(2)_R$ after reality conditions are imposed) that preserves the level set $Q_R = \frac{1}{2i}R\sigma_3$. This undetermined $\mathbb{C}^*$ conjugation matrix can be expressed in terms of a single arbitrary constant $\bar{\theta}_0 \in \mathbb{C}$ as $e^{\frac{i}{2}\bar{\theta}_0\sigma_3}$. As we have argued, the initial $\mathrm{U}(1)_R$ angle $\bar{\theta}_0$ can be specified by the representative $\hat{\gamma}(0,0)$ of the equivalence class $\mathcal{A}(\hat{\gamma}(0,0))$. The relation of $\bar{\theta}_0$ to $\hat{\gamma}(0,0)$ will become clear in section 3.4 when we will extend the target of the Abel map (3.28) topologically by a $\mathbb{C}^*$ factor, turning it into an extended Abel map $\vec{\mathcal{A}} : S^{g+1}(\Sigma) \to J(\Sigma,\infty^\pm)$ that maps bijectively into the generalised Jacobian $J(\Sigma,\infty^\pm)$ to be defined later.

- The function $\Theta_\pm$ is 2×2 matrix valued and its only feature we are interested in for the present purposes is that its (σ,τ)-dependence enters solely through the b-periods of $d\mathcal{Q}(\sigma,\tau)$ in the expression

$$\boldsymbol{\mathcal{A}}(\hat{\gamma}(\sigma,\tau)) = \boldsymbol{\mathcal{A}}(\hat{\gamma}(0,0)) - \int_{\boldsymbol{b}} d\mathcal{Q}(\sigma,\tau).$$

Likewise, it is important to note that the quantity entering in the exponents of expression (3.27) for $j_\pm$ is just (minus) the $\mathcal{B}_{g+1}$-period of $d\mathcal{Q}(\sigma,\tau)$, namely

$$\int_{\infty^-}^{\infty^+} d\mathcal{Q}(\sigma,\tau) = -\int_{\mathcal{B}_{g+1}} d\mathcal{Q}(\sigma,\tau).$$

Moreover, from (3.26) we can also write down a formal expression for the fundamental field g, out of which the $\mathrm{SU}(2)_R$ current $j = -g^{-1}dg = (dg^{-1})g$ is constructed, up to an $\mathrm{SU}(2)_L$ symmetry (or $\mathrm{SL}(2,\mathbb{C})_L$ before imposing reality conditions)

$$g = \sqrt{\det \hat{\Psi}(0)} \cdot \hat{\Psi}(0)^{-1} \in \mathrm{SL}(2,\mathbb{C}).$$

As for the current j above, an explicit representations of the group element g in terms of Riemann θ-functions associated with Σ can also be constructed. Making use of the dual Baker-Akhiezer vector [1] to express $\hat{\Psi}(0)^{-1}$ we find that the components Z_i of g in (2.1) are proportional to the components $\widetilde{\psi}_i^+(0^+)$ of the dual Baker-Akhiezer vector at $P = 0^+$, i.e.

$$Z_1 = Z_1(0,0) \frac{\theta\big(2\pi \int_{\infty^+}^{0^+} \boldsymbol{\omega} - \int_{\boldsymbol{b}} d\mathcal{Q} - \boldsymbol{D}\big)}{\theta\big(\int_{\boldsymbol{b}} d\mathcal{Q} + \boldsymbol{D}\big)} \exp\left(-i \int_{\infty^+}^{0^+} d\mathcal{Q}\right),$$

$$Z_2 = Z_2(0,0) \frac{\theta\big(2\pi \int_{\infty^-}^{0^+} \boldsymbol{\omega} - \int_{\boldsymbol{b}} d\mathcal{Q} - \boldsymbol{D}\big)}{\theta\big(\int_{\boldsymbol{b}} d\mathcal{Q} + \boldsymbol{D}\big)} \exp\left(-i \int_{\infty^-}^{0^+} d\mathcal{Q}\right),$$

where $\boldsymbol{D} = \boldsymbol{\mathcal{A}}(\hat{\gamma}^+(0,0)) + \boldsymbol{\mathcal{K}} \in \mathbb{C}^g$ ($\hat{\gamma}^+(0,0)$ being the dual divisor to $\hat{\gamma}(0,0)$, see (3.47), and $\boldsymbol{\mathcal{K}}$ being the vector of Riemann's constants) is almost arbitrary and $Z_i(0,0) \in \mathbb{C}$ are constants which can be expressed in terms of the algebro-geometric data. Using the property $\hat{\sigma}^* d\mathcal{Q} = -d\mathcal{Q}$ of the differential $d\mathcal{Q}$ where $\hat{\sigma}x^{\pm} = x^{\mp}$ is the hyperelliptic involution of Σ we can rewrite the above expressions in a way that emphasises the linearisation of the motion in the global $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L$ directions, namely[8]

$$Z_1 = Z_1(0,0) \frac{\theta\big(2\pi \int_{\infty^+}^{0^+} \boldsymbol{\omega} - \int_{\boldsymbol{b}} d\mathcal{Q} - \boldsymbol{D}\big)}{\theta\big(\int_{\boldsymbol{b}} d\mathcal{Q} + \boldsymbol{D}\big)} \exp\left(+\frac{i}{2} \int_{\infty^-}^{\infty^+} d\mathcal{Q} - \frac{i}{2} \int_{0^-}^{0^+} d\mathcal{Q}\right), \tag{3.29a}$$

$$Z_2 = Z_2(0,0) \frac{\theta\big(2\pi \int_{\infty^-}^{0^+} \boldsymbol{\omega} - \int_{\boldsymbol{b}} d\mathcal{Q} - \boldsymbol{D}\big)}{\theta\big(\int_{\boldsymbol{b}} d\mathcal{Q} + \boldsymbol{D}\big)} \exp\left(-\frac{i}{2} \int_{\infty^-}^{\infty^+} d\mathcal{Q} - \frac{i}{2} \int_{0^-}^{0^+} d\mathcal{Q}\right). \tag{3.29b}$$

3.2 Extracting data

Because (3.27) is the general solution to the field equations (2.3), its restriction to a given τ-slice, say $\tau = 0$, can be used as a convenient parametrisation of the most general phase-space configuration $j(\sigma) = (j_0(\sigma), j_1(\sigma))$ of the string. Furthermore, since the current (3.27) also satisfies the Virasoro constraints (2.8), the parameters it depends on are independent physical degrees of freedom of the string. In the remainder of the paper we shall therefore use the following parametrisation of the phase-space configuration $j(\sigma)$

$$\begin{aligned} j_{\pm}(\sigma) = e^{\left(\frac{i}{2}\bar{\theta}_0 + \frac{i}{2} n_{g+1}\sigma\right)\sigma_3} &\Theta_{\pm}\left(\boldsymbol{\mathcal{A}}(\hat{\gamma}(0,0)) - \boldsymbol{k}\sigma; \boldsymbol{\mathcal{A}}(\hat{\gamma}(0,0))\right) \\ &\times (i\kappa_{\pm}\sigma_3)\, \Theta_{\pm}\left(\boldsymbol{\mathcal{A}}(\hat{\gamma}(0,0)) - \boldsymbol{k}\sigma; \boldsymbol{\mathcal{A}}(\hat{\gamma}(0,0))\right)^{-1} e^{-\left(\frac{i}{2}\bar{\theta}_0 + \frac{i}{2} n_{g+1}\sigma\right)\sigma_3}, \end{aligned} \tag{3.30}$$

[8]These solutions seem to be closely related to the solutions obtained in an appendix of [6] following the method of [15]. One apparent difference, however, is that the latter are constructed from the Θ-functions of a certain double-cover of the spectral curve Σ considered here. We do not yet understand the precise connection between the two results.

where $\boldsymbol{k} = \int_{\boldsymbol{b}} \frac{dp}{2\pi}$ and $n_{g+1} = \int_{\mathcal{B}_{g+1}} \frac{dp}{2\pi}$ after noting that $d\mathcal{Q}(\sigma, 0) = \frac{\sigma}{2\pi} dp$. This is the analogue of the mode expansion (3.2) for the general phase-space configuration in the flat-space case. Just as one can also extract the parameters of the mode expansion (3.3) from a general phase-space configuration in flat-space, it is possible to extract the divisor $\hat{\gamma}(0,0)$ from a general 'finite-gap' phase-space configuration (3.30) as we now show.

Indeed, the divisor $\hat{\gamma}(0,0)$ of poles of $\boldsymbol{h}(P,0,0)$ can be extracted à la Sklyanin from $\Omega(x) \equiv \Omega(x,0,0)$. Introducing the notation $\boldsymbol{h}_i = \text{res}_{P=\hat{\gamma}_i} \boldsymbol{h}(P,0,0)$ where $\hat{\gamma}(0,0) = \prod_{i=1}^{g+1} \hat{\gamma}_i$, we have

$$\begin{cases} \Omega(x_{\hat{\gamma}_i}) \boldsymbol{h}_i = \Lambda(\hat{\gamma}_i) \boldsymbol{h}_i, \\ \boldsymbol{\alpha} \cdot \boldsymbol{h}_i = 0. \end{cases} \tag{3.31}$$

However, to simplify the forthcoming calculations of Poisson brackets we perform the following similarity transformation on the system of equations (3.31)

$$\boldsymbol{h}_i \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \boldsymbol{h}_i = \tilde{\boldsymbol{h}}_i, \quad \Omega(x_{\hat{\gamma}_i}) \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Omega(x_{\hat{\gamma}_i}) \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \widetilde{\Omega}(x_{\hat{\gamma}_i}), \tag{3.32}$$

so that the system now reads

$$\begin{cases} \widetilde{\Omega}(x_{\hat{\gamma}_i}) \tilde{\boldsymbol{h}}_i = \Lambda(\hat{\gamma}_i) \tilde{\boldsymbol{h}}_i, \\ \left(\tilde{\boldsymbol{h}}_i\right)_1 = 0. \end{cases}$$

The points $\{\hat{\gamma}_i\}_{i=1}^{g+1}$ of the divisor $\hat{\gamma}(0,0)$ are therefore characterised in terms of the components $\widetilde{\mathcal{A}}(x)$ and $\widetilde{\mathcal{B}}(x)$ of

$$\widetilde{\Omega}(x) = \begin{pmatrix} \widetilde{\mathcal{A}}(x) & \widetilde{\mathcal{B}}(x) \\ \widetilde{\mathcal{C}}(x) & \widetilde{\mathcal{D}}(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A}(x) & \mathcal{B}(x) \\ \mathcal{C}(x) & \mathcal{D}(x) \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \tag{3.33}$$

as follows

$$\widetilde{\mathcal{B}}(x_{\hat{\gamma}_i}) = 0, \quad \Lambda(\hat{\gamma}_i) = \widetilde{\mathcal{D}}(x_{\hat{\gamma}_i}) = \widetilde{\mathcal{A}}(x_{\hat{\gamma}_i})^{-1}. \tag{3.34a}$$

Note that $\widetilde{\mathcal{B}}(x)$ actually has infinitely many zeroes but only $g+1$ of them constitute the divisor $\hat{\gamma}(0,0)$, the remaining zeroes being the singular points of the curve Γ. Thus the initial data $\hat{\gamma}(0,0)$ pertaining to the divisor $\hat{\gamma}(\sigma,\tau)$, i.e. to the physical degrees of freedom, can be retrieved from the $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ components of

$$\widetilde{\Omega}(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} P\overleftarrow{\exp} \int_0^{2\pi} d\sigma' \frac{1}{2} \left(\frac{j_+(\sigma')}{1-x} - \frac{j_-(\sigma')}{1+x} \right) \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \tag{3.34b}$$

Equations (3.34a) and (3.34b) will be our way of extracting the initial data pertaining to the physical degrees of freedom from a general field configuration. This is the non-linear analogue of extracting the Fourier coefficients in the flat space case, c.f. equation (3.3).

Note that the matrix from which one reads off the divisor $\hat{\gamma}(0,0)$ isn't exactly the monodromy matrix $\Omega(x)$ itself but instead a similarity transformation of it, namely

$$\widetilde{\Omega}(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Omega(x) \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

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Therefore in appendix B we relate the bracket $\{\widetilde{\Omega}(x) \overset{\otimes}{,} \widetilde{\Omega}(x')\}$ to the bracket (2.22) of monodromy matrices. The result is simply that the matrix $\widetilde{\Omega}(x)$ satisfies exactly the same algebra as the monodromy matrix $\Omega(x)$ itself, and so we shall henceforth only refer to $\widetilde{\Omega}(x)$ since it is the matrix relevant for retrieving the divisor $\hat{\gamma}(0,0)$.

3.3 Poisson brackets of algebro-geometric data

Poisson brackets between the components $\widetilde{\mathcal{A}}(x)$ and $\widetilde{\mathcal{B}}(x)$ of $\widetilde{\Omega}(x)$ can be deduced from (2.22) as is done in appendix C,

$$\left\{\widetilde{\mathcal{A}}(x), \widetilde{\mathcal{A}}(x')\right\} = \left(\widetilde{\mathcal{B}}(x)\widetilde{\mathcal{C}}(x') - \widetilde{\mathcal{B}}(x')\widetilde{\mathcal{C}}(x)\right) \hat{s}(x,x'), \tag{3.35a}$$

$$\left\{\widetilde{\mathcal{A}}(x), \widetilde{\mathcal{B}}(x')\right\} = \left(\widetilde{\mathcal{A}}(x)\widetilde{\mathcal{B}}(x') + \widetilde{\mathcal{A}}(x')\widetilde{\mathcal{B}}(x)\right) \hat{r}(x,x') \tag{3.35b}$$

$$+ \left(\widetilde{\mathcal{A}}(x)\widetilde{\mathcal{B}}(x') + \widetilde{\mathcal{D}}(x')\widetilde{\mathcal{B}}(x)\right) \hat{s}(x,x'), \tag{3.35c}$$

$$\left\{\widetilde{\mathcal{B}}(x), \widetilde{\mathcal{B}}(x')\right\} = 0, \tag{3.35d}$$

where $\hat{r}(x,x')$ and $\hat{s}(x,x')$ are defined as $r(x,x')$ and $s(x,x')$ respectively without the factors of η, i.e. $r(x,x') = \hat{r}(x,x')\eta$ and $s(x,x') = \hat{s}(x,x')\eta$.

In this subsection, we will show that the above relations imply non-trivial Poisson brackets between the complex variables comprising the algebro-geometric data. We will consider the implications of the three relations (3.35) in turn. First we take the limit $x' \to x_{\hat{\gamma}_l}$ of (3.35a). Using (3.34) this gives

$$\{\widetilde{\mathcal{A}}(x), \Lambda(\hat{\gamma}_l)^{-1}\} = \widetilde{\mathcal{B}}(x)\widetilde{\mathcal{C}}(x_{\hat{\gamma}_l})\hat{s}\left(x, x_{\hat{\gamma}_l}\right). \tag{3.36}$$

Taking the limit $x \to x_{\hat{\gamma}_k}$ immediately gives,

$$\{\Lambda(\hat{\gamma}_k)^{-1}, \Lambda(\hat{\gamma}_l)^{-1}\} = 0.$$

We now turn to the Poisson bracket (3.35b). Taking the limit $x \to x_{\hat{\gamma}_l}$ first gets rid of the terms proportional to $\widetilde{\mathcal{B}}(x)$ (using $\widetilde{\mathcal{B}}(x_{\hat{\gamma}_l}) = 0$) and leaves

$$\left\{\widetilde{\mathcal{A}}(x_{\hat{\gamma}_l}), \widetilde{\mathcal{B}}(x')\right\} = \widetilde{\mathcal{A}}(x_{\hat{\gamma}_l})\widetilde{\mathcal{B}}(x')\left(\hat{r}(x_{\hat{\gamma}_l}, x') + \hat{s}(x_{\hat{\gamma}_l}, x')\right).$$

Now using (3.34) we can write $\widetilde{\mathcal{B}}(x') = (x' - x_{\hat{\gamma}_k})\widetilde{\mathcal{B}}_k(x')$ with $\widetilde{\mathcal{B}}_k(x_{\hat{\gamma}_k}) \neq 0$, so that

$$\begin{aligned}(x' - x_{\hat{\gamma}_k})\left\{\widetilde{\mathcal{A}}(x_{\hat{\gamma}_l}), \widetilde{\mathcal{B}}_k(x')\right\} - \left\{\widetilde{\mathcal{A}}(x_{\hat{\gamma}_l}), x_{\hat{\gamma}_k}\right\}\widetilde{\mathcal{B}}_k(x') \\ = \widetilde{\mathcal{A}}(x_{\hat{\gamma}_l})(x' - x_{\hat{\gamma}_k})\widetilde{\mathcal{B}}_k(x')\left(\hat{r}(x_{\hat{\gamma}_l}, x') + \hat{s}(x_{\hat{\gamma}_l}, x')\right),\end{aligned}$$

where

$$\hat{r}(x_{\hat{\gamma}_l}, x') + \hat{s}(x_{\hat{\gamma}_l}, x') = -\frac{2\pi}{\sqrt{\lambda}} \frac{x_{\hat{\gamma}_l}^2 + x'^2 - 2x_{\hat{\gamma}_l}^2 x'^2}{(x_{\hat{\gamma}_l} - x')(1 - x_{\hat{\gamma}_l}^2)(1 - x'^2)} - \frac{2\pi}{\sqrt{\lambda}} \frac{x_{\hat{\gamma}_l} + x'}{(1 - x_{\hat{\gamma}_l}^2)(1 - x'^2)}.$$

It is easy to see that taking the limit $x' \to x_{\hat{\gamma}_k}$ with $k \neq l$ kills everything but the second term on the left hand side, leaving $\{\widetilde{\mathcal{A}}(x_{\hat{\gamma}_l}), x_{\hat{\gamma}_k}\} = 0$, $k \neq l$. Now setting $k = l$ and taking the limit $x' \to x_{\hat{\gamma}_l}$ kills the $\hat{s}$ term leaving $-\left\{\widetilde{\mathcal{A}}(x_{\hat{\gamma}_l}), x_{\hat{\gamma}_l}\right\} = \frac{4\pi}{\sqrt{\lambda}}\widetilde{\mathcal{A}}(x_{\hat{\gamma}_l})\frac{x_{\hat{\gamma}_l}^2}{1-x_{\hat{\gamma}_l}^2}$. Thus, again using (3.34), we have

$$\left\{\Lambda(\hat{\gamma}_l)^{-1}, x_{\hat{\gamma}_k}\right\} = \frac{4\pi}{\sqrt{\lambda}}\Lambda(\hat{\gamma}_l)^{-1}\frac{x_{\hat{\gamma}_l}^2}{x_{\hat{\gamma}_l}^2 - 1}\delta_{kl}.$$

Finally we turn our attention to (3.35d). Again, writing $\widetilde{\mathcal{B}}(x) = (x - x_{\hat{\gamma}_l})\widetilde{\mathcal{B}}_l(x)$ it immediately follows from the third equation (3.35d) that $\left\{x_{\hat{\gamma}_l}, \widetilde{\mathcal{B}}(x')\right\} = 0$ which in turn implies that for all $k, l = 1\ldots, g+1$

$$\{x_{\hat{\gamma}_l}, x_{\hat{\gamma}_k}\} = 0.$$

The algebro-geometric data needed to reconstruct a finite-gap solution is specified by the $2K = 2(g+1)$ complex coordinates, $\{x_{\hat{\gamma}_l}, \Lambda(\hat{\gamma}_l)\}$ for $l = 1, \ldots, g+1$. The results obtained above constitute a complete set of Poisson brackets for these variables. To write these brackets in canonical form we change variables to,

$$z(\hat{\gamma}_l) = x_{\hat{\gamma}_l} + \frac{1}{x_{\hat{\gamma}_l}}, \qquad \mathcal{P}(\hat{\gamma}_l) = \frac{\sqrt{\lambda}}{4\pi}\log\Lambda(\hat{\gamma}_l).$$

Note that $\mathcal{P}(\gamma_l)$ is related to the quasi-momentum at the point $\hat{\gamma}_l$ as $\mathcal{P}(\hat{\gamma}_l) = \frac{i\sqrt{\lambda}}{4\pi}p(\hat{\gamma}_l)$.

In these variables the complete set of Poisson brackets for the algebro-geometric data becomes,

$$\begin{aligned} \{z(\hat{\gamma}_l), z(\hat{\gamma}_m)\} &= 0, \\ \{\mathcal{P}(\hat{\gamma}_l), \mathcal{P}(\hat{\gamma}_m)\} &= 0, \\ \{z(\hat{\gamma}_l), \mathcal{P}(\hat{\gamma}_m)\} &= \delta_{lm}. \end{aligned} \tag{3.37}$$

3.4 Action-angle variables

The change of coordinates to action-angle variables is fairly standard and was reviewed in the case of the internal degrees of freedom of the string in [1]. Here we construct the complete set of action-angle variables starting from the algebro-geometric symplectic form (3.37) on $\mathcal{M}_{\mathbb{C}}^{(2g+2)}$ obtained in the previous section,

$$\hat{\omega}_{2K} = -\frac{\sqrt{\lambda}}{4\pi i}\sum_{i=1}^{g+1}\delta p(\hat{\gamma}_i)\wedge\delta z(\hat{\gamma}_i), \tag{3.38}$$

which is naturally defined on the symmetric product bundle $\mathcal{M}_{\mathbb{C}}^{(2g+2)}$ over $\mathcal{L}$ introduced in (3.24).

3.4.1 Symplectic transformation

It is useful to consider first the universal curve bundle $\mathcal{N}$ over the leaf $\mathcal{L}$

$$\Sigma \to \mathcal{N} \to \mathcal{L},$$

whose fibre over every point of the base $\mathcal{L}$ is the corresponding curve Σ. Recall from section 3.1.1 that the $\{S_i\}_{i=1}^g$ and R defined in (3.15) form a set of coordinates on the base $\mathcal{L}$, and note that z can be taken as a coordinate along the fibre. Denote by δ the exterior derivative on the total space $\mathcal{N}$ and consider the differential $\delta\tilde{\alpha}$ of $\tilde{\alpha} = -\frac{\sqrt{\lambda}}{4\pi i} p dz$ on $\mathcal{N}$

$$-\frac{\sqrt{\lambda}}{4\pi i}\delta p \wedge dz = \delta\tilde{\alpha} = \sum_{i=1}^{g} \delta S_i \wedge \partial_{S_i}\tilde{\alpha} + \frac{1}{2}\delta R \wedge \partial_{\frac{R}{2}}\tilde{\alpha}. \tag{3.39}$$

The coordinates $\{S_i\}_{i=1}^g$ and R can be expressed in terms as the appropriately normalised $\boldsymbol{a}$-periods and residue at ∞^+ of the differential $\tilde{\alpha}$,

$$S_i = \frac{1}{2\pi}\int_{a_i}\tilde{\alpha}, \; i = 1, \ldots, g, \quad \frac{R}{2} = -\frac{1}{2\pi}\int_{c_{\infty^+}}\tilde{\alpha}, \tag{3.40}$$

where c_{∞^+} is a counter-clockwise cycle around the point $\infty^+ \in \Sigma$. Now the key observation is that although $\tilde{\alpha}$ is neither single-valued nor holomorphic on Σ, the ambiguities in its definition are constant along the leaf $\mathcal{L}$ and its pole parts are constant except for those around $\infty^\pm \in \Sigma$ which are proportional to R. It follows therefore from (3.40) that

$$\partial_{S_i}\tilde{\alpha} = 2\pi\omega_i, \; i = 1, \ldots, g, \quad \partial_{\frac{R}{2}}\tilde{\alpha} = -2\pi\omega_\infty,$$

where ω_∞ is the normalised Abelian differential of the third kind with simple poles at $\infty^\pm$ with residues $\pm\frac{1}{2\pi i}$ respectively. Therefore (3.39) simplifies to

$$\delta\tilde{\alpha} = \sum_{i=1}^{g}\delta S_i \wedge 2\pi\omega_i - \frac{1}{2}\delta R \wedge 2\pi\omega_\infty.$$

This differential $\delta\tilde{\alpha}$ living on $\mathcal{N}$ can be used to define the symplectic form $\hat{\omega}_{2K}$ on $\mathcal{M}_{\mathbb{C}}^{(2g+2)}$ by the following expression which is symmetric in the points $\hat{\gamma}_j \in \Sigma$, $j = 1, \ldots, g+1$,

$$\hat{\omega}_{2K} = \sum_{j=1}^{g+1}\delta\tilde{\alpha}(\hat{\gamma}_j) = \sum_{i=1}^{g}\delta S_i \wedge 2\pi\left(\sum_{j=1}^{g+1}\omega_i(\hat{\gamma}_j)\right) - \frac{1}{2}\delta R \wedge 2\pi\left(\sum_{j=1}^{g+1}\omega_\infty(\hat{\gamma}_j)\right).$$

However, the $(g+1)^{\mathrm{st}}$ symmetric product $S^{g+1}(\Sigma) = \Sigma^{g+1}/S_{g+1}$ of the curve Σ is isomorphic to the $(g+1)$-dimensional *generalised Jacobian*[9] $J(\Sigma, \infty^\pm)$ of the curve Σ with two punctures at $\infty^\pm$ via the extended Abel map

$$\begin{aligned}
&\vec{\mathcal{A}} : S^{g+1}(\Sigma) \to J(\Sigma, \infty^\pm) \\
&D = \prod_{j=1}^{g+1} P_j \mapsto (\boldsymbol{\mathcal{A}}(D), \mathcal{A}_{g+1}(D)) = \left(2\pi\sum_{j=1}^{g+1}\int_{P_0}^{P_j}\boldsymbol{\omega}, -2\pi\sum_{j=1}^{g+1}\int_{P_0}^{P_j}\omega_\infty\right),
\end{aligned} \tag{3.41}$$

[9]The generalised Jacobian is an extension of the standard notion of a Jacobian to singular surfaces (see for example [22] and references therein) which can be thought of as limits of regular Riemann surfaces. In the present case the singular curve is $\Sigma/\{\infty^\pm\}$ (with a degenerated handle at ∞) and its generalised Jacobian $J(\Sigma, \infty^\pm)$ is topologically equivalent to the Cartesian product $J(\Sigma) \times \mathbb{C}^*$ of the standard g-dimensional Jacobian $J(\Sigma)$ with the cylinder $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

where $P_0 \in \Sigma$ is arbitrary. The first g components of this map make up the usual Abel map $\boldsymbol{\mathcal{A}} : S^{g+1}(\Sigma) \to J(\Sigma)$ defined in (3.28) on divisors of degree $g+1$. Whereas the Abel map (3.28) was surjective, the extended Abel map (3.41) is bijective.

So if we define (complex) coordinates on $J(\Sigma, \infty^{\pm})$ as

$$\boldsymbol{\theta} = \boldsymbol{\mathcal{A}}(\hat{\gamma}(0,0)), \quad \bar{\theta} = \mathcal{A}_{g+1}(\hat{\gamma}(0,0)) \tag{3.42}$$

and identify $\mathcal{M}_{\mathbb{C}}^{(2g+2)}$ with the extended Jacobian bundle $J(\Sigma, \infty^{\pm}) \to \mathcal{J}(\Sigma) \to \mathcal{L}$ using the extended Abel map (3.41) then

$$\hat{\omega}_{2K} = \sum_{i=1}^{g} \delta S_i \wedge \delta\theta_i + \frac{1}{2}\delta R \wedge \delta\bar{\theta}. \tag{3.43}$$

It will be convenient to consider a slightly different set of action-angle variables first proposed in [1] in which the filling fractions (3.16) play the role of the action variables. For this we rewrite (3.43) as follows

$$\begin{aligned} \hat{\omega}_{2K} &= \sum_{i=1}^{g} \delta S_i \wedge \left(\delta\theta_i - \delta\bar{\theta}\right) + \left(\frac{1}{2}\delta R + \sum_{i=1}^{g} \delta S_i\right) \wedge \delta\bar{\theta} \\ &= \sum_{i=1}^{g} \delta S_i \wedge \delta\left(\theta_i - \bar{\theta}\right) + \delta\left(\frac{L-R}{2} - \sum_{i=1}^{g} S_i\right) \wedge \delta\left(-\bar{\theta}\right), \end{aligned} \tag{3.44}$$

where in the second line we use the fact that $\delta L = 0$ since L is fixed along the leaf $\mathcal{L}$ under consideration. Now recalling the definition of the $K = g+1$ filling fractions $\{\mathcal{S}_I\}_{I=1}^{K}$ introduced in (3.16) and introducing a new set of angle variables $\{\varphi_I\}_{I=1}^{K}$ related to the $\boldsymbol{\theta}, \bar{\theta}$ by [1]

$$\varphi_I = \theta_i - \bar{\theta} \quad \text{for } I = i = 1, \ldots, g = K-1, \qquad \varphi_K = -\bar{\theta},$$

then equation (3.44) reads

$$\hat{\omega}_{2K} = \sum_{I=1}^{K} \delta\mathcal{S}_I \wedge \delta\varphi_I. \tag{3.45}$$

3.4.2 Reality conditions

The reality of the action variables (3.40)

$$\bar{S}_i = S_i, \; i = 1, \ldots, g, \quad \bar{R} = R,$$

follows immediately [1] from the reality conditions $\overline{\hat{\tau}^*\tilde{\alpha}} = -\tilde{\alpha}, \hat{\tau}\boldsymbol{a} = -\boldsymbol{a}$ and $\hat{\tau}c_{\infty^+} = -c_{\infty^+}$ on the 1-form $\tilde{\alpha}$, the $\boldsymbol{a}$-periods and the cycle c_{∞^+}.

Obtaining real angle variables (3.42) is slightly more involved. In fact, as defined in (3.42) the angles $\boldsymbol{\theta}, \bar{\theta}$ are not real but can be made real after substraction of a constant in each case, which does not affect the result (3.43) of the previous section. We start by recalling from [1] how to obtain real angles $\boldsymbol{\theta}$.

The equivalence class $[\hat{\gamma}(0,0)]$ of the degree $g+1$ divisor $\hat{\gamma}(0,0)$ satisfies a simple reality condition [1], namely

$$\hat{\tau}\hat{\gamma}(0,0) \sim \hat{\gamma}^{+}(0,0), \tag{3.46}$$

where $\hat{\gamma}^+(0,0)$ is the dual divisor to $\hat{\gamma}(0,0)$. The dual divisor is of degree $g+1$ and can be defined up to equivalence by the relation

$$\hat{\gamma}(0,0) \cdot \hat{\gamma}^+(0,0) \sim Z \cdot (\infty^+)^2 \cdot (\infty^-)^2, \tag{3.47}$$

where Z is the canonical class, i.e. the divisor of any Abelian differential (the ratio of any two Abelian differential is a meromorphic function and so their divisors are equivalent). Putting equations (3.46) and (3.47) together, the reality condition on $[\hat{\gamma}(0,0)]$ can be expressed as follows

$$\hat{\gamma}(0,0) \cdot \hat{\tau}\hat{\gamma}(0,0) \sim Z \cdot (\infty^+)^2 \cdot (\infty^-)^2. \tag{3.48}$$

If the base point P_0 of the Abel map is chosen to be real, i.e. such that $\hat{\tau} P_0 = P_0$, then the reality condition on the Abel map reads $\boldsymbol{\mathcal{A}}(\hat{\tau} D) = -\overline{\boldsymbol{\mathcal{A}}(D)}$. It follows then from (3.48) that

$$2 \text{ Im } \boldsymbol{\mathcal{A}}(\hat{\gamma}(0,0)) = \boldsymbol{\mathcal{A}}\left(Z \cdot (\infty^+)^2 \cdot (\infty^-)^2\right).$$

This yields the reality condition on the first g components $\boldsymbol{\mathcal{A}}(\hat{\gamma}(0,0))$ of $\vec{\mathcal{A}}(\hat{\gamma}(0,0))$. The angle coordinates $\boldsymbol{\theta}$ are rendered real after the following redefinition

$$\boldsymbol{\theta} = \boldsymbol{\mathcal{A}}(\hat{\gamma}(0,0)) - \frac{1}{2}\boldsymbol{\mathcal{A}}(Z \cdot (\infty^+)^2 \cdot (\infty^-)^2) \in \text{Re } J(\Sigma). \tag{3.49}$$

We now turn to the reality of the angle $\bar{\theta}$. For this we show that under a change of representative of the class $[\hat{\gamma}(0,0)]$ which is such that the reality condition $|W|^2 = 1$ on (3.19) is satisfied, the corresponding change $\Delta\bar{\theta}$ in the angle $\bar{\theta}$ is real. It would follow from this that representatives of $[\hat{\gamma}(0,0)]$ which give rise to real solutions are mapped under $\mathcal{A}_{g+1}$ to a real subspace of $\mathbb{C}$, i.e. $\mathcal{A}_{g+1}(\hat{\gamma}(0,0)) - \mathcal{C} \in \mathbb{R}$ for some constant $\mathcal{C} \in \mathbb{C}$.

Recall the function $f(P,0,0) = Wh_1(P,0,0) - W^{-1}h_2(P,0,0)$ introduced in (3.18) which has poles at $\hat{\gamma}(0,0)$ and zeroes at the equivalent divisor $\hat{\gamma}'(0,0) = \prod_{i=1}^{g+1} \hat{\gamma}'_i \sim \hat{\gamma}(0,0)$, and consider the differential $df/f = d\log f$. Its only poles are at $\hat{\gamma}(0,0)$ with residues -1 and at $\hat{\gamma}'(0,0)$ with residues $+1$. For an arbitrary pair of points $P, Q \in \Sigma$, let us denote by ω_{PQ} the unique normalised (vanishing $\boldsymbol{a}$-periods) Abelian differential of the third kind with simple poles at P and Q with residues $+1$ and -1 there respectively. Then it follows that

$$\frac{df}{f} - \sum_{j=1}^{g+1} \omega_{\hat{\gamma}'_j \hat{\gamma}_j} = \sum_{i=1}^{g} c_i \omega_i, \tag{3.50}$$

for some constants $c_i \in \mathbb{C}$. Taking the $\mathcal{B}_{g+1}$-period of the last equation yields

$$\int_{\infty^-}^{\infty^+} \frac{df}{f} - \sum_{j=1}^{g+1} \int_{\infty^-}^{\infty^+} \omega_{\hat{\gamma}'_j \hat{\gamma}_j} = \sum_{i=1}^{g} c_i \int_{\infty^-}^{\infty^+} \omega_i. \tag{3.51}$$

Taking the $\boldsymbol{a}$-periods of the equation (3.50) on the other hand gives the constants c_i,

$$c_i = \int_{a_i} \frac{df}{f} = \int_{a_i} d\log f = 2\pi i m_i, \quad m_i \in \mathbb{Z}.$$

Now using the Riemann bilinear identities it is straightforward to show that

$$\int_{\infty^-}^{\infty^+} \omega_{\hat{\gamma}'_j \hat{\gamma}_j} = 2\pi i \int_{\hat{\gamma}_j}^{\hat{\gamma}'_j} \omega_\infty, \qquad \int_{\infty^-}^{\infty^+} \omega_i = \int_{b_i} \omega_\infty, \tag{3.52}$$

and so plugging this back into (3.51) yields the following

$$\int_{\infty^-}^{\infty^+} \frac{df}{f} - 2\pi i \sum_{j=1}^{g+1} \int_{\hat{\gamma}_j}^{\hat{\gamma}'_j} \omega_\infty = 2\pi i \sum_{i=1}^{g} m_i \int_{b_i} \omega_\infty.$$

Referring back to the definition of the extended Abel map (3.41) we recognise the second term on the left hand side of the last expression as the difference between the $(g+1)^{\mathrm{st}}$ components of the Abel maps of the divisors $\hat{\gamma}'(0,0)$ and $\hat{\gamma}(0,0)$,

$$\Delta\bar{\theta} = \mathcal{A}_{g+1}(\hat{\gamma}'(0,0)) - \mathcal{A}_{g+1}(\hat{\gamma}(0,0)) = i \int_{\infty^-}^{\infty^+} \frac{df}{f} + 2\pi \sum_{i=1}^{g} m_i \int_{b_i} \omega_\infty. \tag{3.53}$$

So to show that $\Delta\bar{\theta} \in \mathbb{R}$ it suffices to show the right hand side of (3.53) is real. Consider the first term, which using the limits $f(\infty^\pm) = W^{\pm 1}$ can be simplified as

$$\int_{\infty^-}^{\infty^+} \frac{df}{f} = \int_{\infty^-}^{\infty^+} d\log f = \log\left(\frac{f(\infty^+)}{f(\infty^-)}\right) = 2\log W.$$

This last expression holds as an equality only on $\mathbb{C}/2\pi i\mathbb{Z}$. We now make use of the reality condition $|W|^2 = 1$ on the residual symmetry (3.19), which can be rewritten $\overline{W} = W^{-1}$, and deduce that

$$i \int_{\infty^-}^{\infty^+} \frac{df}{f} \in \mathbb{R}/2\pi\mathbb{Z}.$$

Furthermore, using (3.52) and the reality conditions $\overline{\hat{\tau}^* \boldsymbol{\omega}} = -\boldsymbol{\omega}, \hat{\tau}\mathcal{B}_{g+1} = \mathcal{B}_{g+1} - \sum_{k=1}^{g} a_k$ one can show that

$$\overline{\int_{b_i} \omega_\infty} = -\int_{b_i} \omega_\infty + 1,$$

so after imposing the reality constraint $|W|^2 = 1$, equation (3.53) implies that

$$\Delta\bar{\theta} \in \mathbb{R}/2\pi\mathbb{Z}. \tag{3.54}$$

As we have already remarked, it follows now from (3.54) that the image under $\mathcal{A}_{g+1}$ of the representatives of $[\hat{\gamma}(0,0)]$ which give rise to real solutions forms a real subspace of $\mathbb{C}$, i.e.

$$\bar{\theta} = \mathcal{A}_{g+1}(\hat{\gamma}(0,0)) - \mathcal{C} \in \mathbb{R}/2\pi\mathbb{Z}. \tag{3.55}$$

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A. Algebra of transition matrices

We start with the Poisson bracket (2.17) between two $J_1(\sigma, x)$ matrices for a non-ultralocal system in the $(r-s)$-matrix formalism introduced by Maillet [20] which can be conveniently rewritten as

$$\begin{aligned}\left\{J_1(\sigma,x) \overset{\otimes}{,} J_1(\sigma',x')\right\} = &\left[r(\sigma,x,x'), J_1(\sigma,x)\otimes\mathbf{1} + \mathbf{1}\otimes J_1(\sigma',x')\right]\delta(\sigma-\sigma') \\ &- \left[s(\sigma,x,x'), J_1(\sigma,x)\otimes\mathbf{1} - \mathbf{1}\otimes J_1(\sigma',x')\right]\delta(\sigma-\sigma') \\ &- \left(r(\sigma,x,x') + s(\sigma,x,x') - r(\sigma',x,x') + s(\sigma',x,x')\right)\delta'(\sigma-\sigma'),\end{aligned} \tag{A.1}$$

using the identity $(f(\sigma) - f(\sigma'))\,\delta'(\sigma-\sigma') = -f'(\sigma)\delta(\sigma-\sigma')$ valid for any function f. Now the transition matrix

$$T(\sigma_1, \sigma_2, x) = P\overleftarrow{\exp}\int_{\sigma_2}^{\sigma_1} d\sigma J_1(\sigma, x),$$

is the unique solution to the following differential equation with boundary condition

$$\frac{\partial T}{\partial \sigma_1}(\sigma_1,\sigma_2,x) = J_1(\sigma_1,x)T(\sigma_1,\sigma_2,x), \qquad T(\sigma_2,\sigma_2,x) = \mathbf{1}. \tag{A.2}$$

It also satisfies the following differential equation with the same boundary condition

$$\frac{\partial T}{\partial \sigma_2}(\sigma_1,\sigma_2,x) = -T(\sigma_1,\sigma_2,x)J_1(\sigma_2,x), \qquad T(\sigma_1,\sigma_1,x) = \mathbf{1}.$$

The variation of the system (A.2) gives

$$\frac{\partial \delta T}{\partial \sigma_1}(\sigma_1,\sigma_2,x) = \delta J_1(\sigma_1,x)T(\sigma_1,\sigma_2,x) + J_1(\sigma_1,x)\delta T(\sigma_1,\sigma_2,x), \qquad \delta T(\sigma_1,\sigma_1,x) = 0,$$

of which the unique solution is easily seen to be

$$\begin{aligned}\delta T(\sigma_1,\sigma_2,x) &= \int_{\sigma_2}^{\sigma_1} d\sigma T(\sigma_1,\sigma,x)\delta J_1(\sigma,x)T(\sigma,\sigma_2,x), \\ &= \int_0^{2\pi} d\sigma \epsilon(\sigma_1-\sigma_2)\chi(\sigma;\sigma_1,\sigma_2)T(\sigma_1,\sigma,x)\delta J_1(\sigma,x)T(\sigma,\sigma_2,x),\end{aligned} \tag{A.3}$$

where $\epsilon(\sigma) = \mathrm{sign}(\sigma)$ is the usual sign function and $\chi(\sigma;\sigma_1,\sigma_2)$ is the characteristic function of the interval between σ_1 and σ_2.

Now given the Poisson bracket of the system

$$\left\{A \overset{\otimes}{,} B\right\} = \int d\sigma \left(\frac{\delta A}{\delta q^a(\sigma)} \otimes \frac{\delta B}{\delta \pi^a(\sigma)} - \frac{\delta A}{\delta \pi^a(\sigma)} \otimes \frac{\delta B}{\delta q^a(\sigma)}\right), \tag{A.4}$$

one can relate the bracket of transition matrices to the bracket of currents (A.1) using (A.3)

$$\begin{aligned}\left\{T(\sigma_1,\sigma_2,x) \overset{\otimes}{,} T(\sigma_1',\sigma_2',x')\right\} = &\int_{\sigma_2}^{\sigma_1} d\sigma \int_{\sigma_2'}^{\sigma_1'} d\sigma' \left(T(\sigma_1,\sigma,x)\otimes T(\sigma_1',\sigma',x')\right) \\ &\times \left\{J_1(\sigma,x) \overset{\otimes}{,} J_1(\sigma',x')\right\}\left(T(\sigma,\sigma_2,x)\otimes T(\sigma',\sigma_2',x')\right).\end{aligned} \tag{A.5}$$

Now plugging (A.1) into this expression, one finds after a bit of algebra

$$\{T(\sigma_1,\sigma_2,x) \overset{\otimes}{,} T(\sigma_1',\sigma_2',x')\} = \int_0^{2\pi} d\sigma \int_0^{2\pi} d\sigma' \chi(\sigma;\sigma_1,\sigma_2)\chi(\sigma';\sigma_1',\sigma_2')\epsilon(\sigma_1-\sigma_2)\epsilon(\sigma_1'-\sigma_2')$$
$$\times\Big[\frac{\partial}{\partial\sigma}\Big(T(\sigma_1,\sigma,x)\otimes T(\sigma_1',\sigma',x')(r(\sigma,x,x')-s(\sigma,x,x'))\,T(\sigma,\sigma_2,x)\otimes T(\sigma',\sigma_2',x')\delta(\sigma-\sigma')\Big)$$
$$+\frac{\partial}{\partial\sigma'}\Big(T(\sigma_1,\sigma,x)\otimes T(\sigma_1',\sigma',x')\Big(r(\sigma,x,x')+s(\sigma,x,x')\Big)T(\sigma,\sigma_2,x)\otimes T(\sigma',\sigma_2',x')\delta(\sigma-\sigma')\Big)\Big].$$

Integrating by parts and using the identity $-\frac{\partial}{\partial\sigma}\chi(\sigma;\sigma_1,\sigma_2)\epsilon(\sigma_1-\sigma_2) = \delta(\sigma-\sigma_1)-\delta(\sigma-\sigma_2)$ we obtain

$$\begin{aligned}\big\{T(\sigma_1,\sigma_2,x) \overset{\otimes}{,} T(\sigma_1',\sigma_2',x')\big\} \;=\; &+\epsilon(\sigma_1'-\sigma_2')\chi(\sigma;\sigma_1',\sigma_2') \times T(\sigma_1,\sigma,x)\otimes T(\sigma_1',\sigma,x') \\ &\Big(r(\sigma,x,x')-s(\sigma,x,x')\Big)T(\sigma,\sigma_2,x)\otimes T(\sigma,\sigma_2',x')\,\Big|^{\sigma=\sigma_1}_{\sigma=\sigma_2} \\ &+\epsilon(\sigma_1-\sigma_2)\chi(\sigma;\sigma_1,\sigma_2)\times T(\sigma_1,\sigma,x)\otimes T(\sigma_1',\sigma,x') \\ &\Big(r(\sigma,x,x')+s(\sigma,x,x')\Big)\,T(\sigma,\sigma_2,x)\otimes T(\sigma,\sigma_2',x')\,\Big|^{\sigma=\sigma_1'}_{\sigma=\sigma_2'}\end{aligned} \tag{A.6}$$

B. SL$(2,\mathbb{C})$-invariance of $\{\Omega \overset{\otimes}{,} \Omega\}$

In this appendix we wish to find how the Poisson bracket $\{\Omega\overset{\otimes}{,}\Omega\}$ transforms under a general similarity transformation of $\Omega(x)\to\widetilde{\Omega}(x) = U^{-1}\Omega(x)U$, $U\in\mathrm{SL}(2,\mathbb{C})$. Using the shorthand notation $\overset{1}{A} = A\otimes\mathbf{1}$, $\overset{2}{A} = \mathbf{1}\otimes A$ we can write

$$\begin{aligned}\Big\{\overset{1}{\widetilde{\Omega}}(x),\overset{2}{\widetilde{\Omega}}(x')\Big\} &= \Big\{\overset{1}{U}{}^{-1}\overset{1}{\Omega}(x)\overset{1}{U},\overset{2}{U}{}^{-1}\overset{2}{\Omega}(x')\overset{2}{U}\Big\} = \overset{1}{U}{}^{-1}\overset{2}{U}{}^{-1}\Big\{\overset{1}{\Omega}(x),\overset{2}{\Omega}(x')\Big\}\overset{1}{U}\overset{2}{U} \\ &= \overset{1}{U}{}^{-1}\overset{2}{U}{}^{-1}\left(\Big[\overset{12}{r}(x,x'),\overset{1}{\Omega}(x)\overset{2}{\Omega}(x')\Big]+\overset{1}{\Omega}(x)\overset{12}{s}(x,x')\overset{2}{\Omega}(x')-\overset{2}{\Omega}(x')\overset{12}{s}(x,x')\overset{1}{\Omega}(x)\right)\overset{1}{U}\overset{2}{U} \\ &= \Big[\overset{12}{\widetilde{r}}(x,x'),\overset{1}{\widetilde{\Omega}}(x)\overset{2}{\widetilde{\Omega}}(x')\Big]+\overset{1}{\widetilde{\Omega}}(x)\overset{12}{\widetilde{s}}(x,x')\overset{2}{\widetilde{\Omega}}(x')-\overset{2}{\widetilde{\Omega}}(x')\overset{12}{\widetilde{s}}(x,x')\overset{1}{\widetilde{\Omega}}(x),\end{aligned}$$

where $\widetilde{r}(x,x') = U^{-1}\otimes U^{-1}r(x,x')U\otimes U$ and $\widetilde{s}(x,x') = U^{-1}\otimes U^{-1}s(x,x')U\otimes U$. Now since $r(x,x')$ and $s(x,x')$ are both proportional to $\eta = -t^a\otimes t^a$, we can compute the transformations of $r(x,x')$ and $s(x,x')$ simultaneously by considering

$$\big(U^{-1}\otimes U^{-1}\big)\,\eta\,(U\otimes U) = \big(U^{-1}\otimes\mathbf{1}\big)\big(\mathbf{1}\otimes U^{-1}\big)\,\eta\,(U\otimes\mathbf{1})\,(\mathbf{1}\otimes U)\,.$$

Considering an infinitesimal transformation $U = e^\alpha \sim \mathbf{1}+\alpha+O(\alpha^2)$, $\alpha\in\mathfrak{sl}(2,\mathbb{C})$, one finds straightforwardly that

$$((\mathbf{1}-\alpha)\otimes(\mathbf{1}-\alpha))\,\eta\,((\mathbf{1}+\alpha)\otimes(\mathbf{1}+\alpha)) \sim \eta + O(\alpha^2).$$

Therefore η is invariant under infinitesimal similarity transformations. It follows then that $\widetilde{r}(x,x') = r(x,x')$ and $\widetilde{s}(x,x') = s(x,x')$, so that the (weak) bracket $\{\Omega\overset{\otimes}{,}\Omega\}$ ends up being invariant under similarity transformations as well, namely the same bracket (2.22) holds for the transformed monodromy matrix $\widetilde{\Omega}(x)$

$$\begin{aligned}\left\{\widetilde{\Omega}(x) \overset{\otimes}{,} \widetilde{\Omega}(x')\right\} &= [r(x,x'), \widetilde{\Omega}(x)\otimes\widetilde{\Omega}(x')] \\ &\quad + \left(\widetilde{\Omega}(x)\otimes\mathbf{1}\right) s(x,x') \left(\mathbf{1}\otimes\widetilde{\Omega}(x')\right) \\ &\quad - \left(\mathbf{1}\otimes\widetilde{\Omega}(x')\right) s(x,x') \left(\widetilde{\Omega}(x)\otimes\mathbf{1}\right).\end{aligned} \tag{B.1}$$

C. Algebra of $\widetilde{\mathcal{A}}(x)$ and $\widetilde{\mathcal{B}}(x)$ components

Let us express the right hand side of (B.1) in terms of the components (3.33) of $\widetilde{\Omega}(x)$. We have

$$\begin{aligned}\widetilde{\Omega}(x)\otimes\widetilde{\Omega}(x') &= \begin{pmatrix} \widetilde{\mathcal{A}}(x)\widetilde{\Omega}(x') & \widetilde{\mathcal{B}}(x)\widetilde{\Omega}(x') \\ \widetilde{\mathcal{C}}(x)\widetilde{\Omega}(x') & \widetilde{\mathcal{D}}(x)\widetilde{\Omega}(x') \end{pmatrix}, \\ \mathbf{1}\otimes\widetilde{\Omega}(x') &= \begin{pmatrix} \widetilde{\Omega}(x') & 0 \\ 0 & \widetilde{\Omega}(x') \end{pmatrix}, \widetilde{\Omega}(x)\otimes\mathbf{1} = \begin{pmatrix} \widetilde{\mathcal{A}}(x)\mathbf{1} & \widetilde{\mathcal{B}}(x)\mathbf{1} \\ \widetilde{\mathcal{C}}(x)\mathbf{1} & \widetilde{\mathcal{D}}(x)\mathbf{1} \end{pmatrix},\end{aligned}$$

and since the matrices $r(x,x')$ and $s(x,x')$ are both proportional to η with

$$\eta = -t^a\otimes t^a = \frac{1}{2}\sigma_a\otimes\sigma_a = \frac{1}{2}\begin{pmatrix} \sigma_3 & \sigma_1 - i\sigma_2 \\ \sigma_1 + i\sigma_2 & -\sigma_3 \end{pmatrix},$$

where $t^a = \frac{i}{\sqrt{2}}\sigma_a$ in the $\mathfrak{su}(2)$ case (σ_a being the Pauli matrices), we need to compute the following quantities

$$\begin{aligned}&\eta\widetilde{\Omega}(x)\otimes\widetilde{\Omega}(x') \\ &\quad = \frac{1}{2}\begin{pmatrix} \widetilde{\mathcal{A}}(x)\sigma_3\widetilde{\Omega}(x') + \widetilde{\mathcal{C}}(x)(\sigma_1 - i\sigma_2)\widetilde{\Omega}(x') & \widetilde{\mathcal{B}}(x)\sigma_3\widetilde{\Omega}(x') + \widetilde{\mathcal{D}}(x)(\sigma_1 - i\sigma_2)\widetilde{\Omega}(x') \\ \widetilde{\mathcal{A}}(x)(\sigma_1 + i\sigma_2)\widetilde{\Omega}(x') - \widetilde{\mathcal{C}}(x)\sigma_3\widetilde{\Omega}(x') & \widetilde{\mathcal{B}}(x)(\sigma_1 + i\sigma_2)\widetilde{\Omega}(x') - \widetilde{\mathcal{D}}(x)\sigma_3\widetilde{\Omega}(x') \end{pmatrix},\end{aligned}$$

$$\begin{aligned}&\widetilde{\Omega}(x)\otimes\widetilde{\Omega}(x')\eta \\ &\quad = \frac{1}{2}\begin{pmatrix} \widetilde{\mathcal{A}}(x)\widetilde{\Omega}(x')\sigma_3 + \widetilde{\mathcal{B}}(x)\widetilde{\Omega}(x')(\sigma_1 + i\sigma_2) & \widetilde{\mathcal{A}}(x)\widetilde{\Omega}(x')(\sigma_1 - i\sigma_2) - \widetilde{\mathcal{B}}(x)\widetilde{\Omega}(x')\sigma_3 \\ \widetilde{\mathcal{C}}(x)\widetilde{\Omega}(x')\sigma_3 + \widetilde{\mathcal{D}}(x)\widetilde{\Omega}(x')(\sigma_1 + i\sigma_2) & \widetilde{\mathcal{C}}(x)\widetilde{\Omega}(x')(\sigma_1 - i\sigma_2) - \widetilde{\mathcal{D}}(x)\widetilde{\Omega}(x')\sigma_3 \end{pmatrix},\end{aligned}$$

$$\begin{aligned}&\left(\widetilde{\Omega}(x)\otimes\mathbf{1}\right)\eta\left(\mathbf{1}\otimes\widetilde{\Omega}(x')\right) \\ &\quad = \frac{1}{2}\begin{pmatrix} \widetilde{\mathcal{A}}(x)\sigma_3\widetilde{\Omega}(x') + \widetilde{\mathcal{B}}(x)(\sigma_1 + i\sigma_2)\widetilde{\Omega}(x') & \widetilde{\mathcal{A}}(x)(\sigma_1 - i\sigma_2)\widetilde{\Omega}(x') - \widetilde{\mathcal{B}}(x)\sigma_3\widetilde{\Omega}(x') \\ \widetilde{\mathcal{C}}(x)\sigma_3\widetilde{\Omega}(x') + \widetilde{\mathcal{D}}(x)(\sigma_1 + i\sigma_2)\widetilde{\Omega}(x') & \widetilde{\mathcal{C}}(x)(\sigma_1 - i\sigma_2)\widetilde{\Omega}(x') - \widetilde{\mathcal{D}}(x)\sigma_3\widetilde{\Omega}(x') \end{pmatrix},\end{aligned}$$

$$\begin{aligned}&\left(\mathbf{1}\otimes\widetilde{\Omega}(x')\right)\eta\left(\widetilde{\Omega}(x)\otimes\mathbf{1}\right) \\ &\quad = \frac{1}{2}\begin{pmatrix} \widetilde{\mathcal{A}}(x)\widetilde{\Omega}(x')\sigma_3 + \widetilde{\mathcal{C}}(x)\widetilde{\Omega}(x')(\sigma_1 - i\sigma_2) & \widetilde{\mathcal{B}}(x)\widetilde{\Omega}(x')\sigma_3 + \widetilde{\mathcal{D}}(x)\widetilde{\Omega}(x')(\sigma_1 - i\sigma_2) \\ \widetilde{\mathcal{A}}(x)\widetilde{\Omega}(x')(\sigma_1 + i\sigma_2) - \widetilde{\mathcal{C}}(x)\widetilde{\Omega}(x')\sigma_3 & \widetilde{\mathcal{B}}(x)\widetilde{\Omega}(x')(\sigma_1 + i\sigma_2) - \widetilde{\mathcal{D}}(x)\widetilde{\Omega}(x')\sigma_3 \end{pmatrix}.\end{aligned}$$

We can read off from this and equation (B.1) the Poisson brackets between various components of $\widetilde{\Omega}(x)$, but we are particular interested in the $\widetilde{\mathcal{A}}(x)$ and $\widetilde{\mathcal{B}}(x)$ components which are given by

$$\begin{aligned}\left\{\widetilde{\mathcal{A}}(x),\widetilde{\mathcal{A}}(x')\right\} &= \left\{\widetilde{\Omega}_{11}(x),\widetilde{\Omega}_{11}(x')\right\} = \left\{\widetilde{\Omega}(x) \stackrel{\otimes}{,} \widetilde{\Omega}(x')\right\}_{11,11}\\ &= \left(\widetilde{\mathcal{B}}(x)\widetilde{\mathcal{C}}(x') - \widetilde{\mathcal{B}}(x')\widetilde{\mathcal{C}}(x)\right)\hat{s}(x,x'),\end{aligned}$$

where $\hat{s}(x,x') = -\frac{2\pi}{\sqrt{\lambda}}\frac{x+x'}{(1-x^2)(1-x'^2)}$ is $s(x,x')$ without the matrix factor η, as well as

$$\begin{aligned}\left\{\widetilde{\mathcal{A}}(x),\widetilde{\mathcal{B}}(x')\right\} &= \left\{\widetilde{\Omega}_{11}(x),\widetilde{\Omega}_{12}(x')\right\} = \left\{\widetilde{\Omega}(x) \stackrel{\otimes}{,} \widetilde{\Omega}(x')\right\}_{11,12}\\ &= \left(\widetilde{\mathcal{A}}(x)\widetilde{\mathcal{B}}(x') + \widetilde{\mathcal{A}}(x')\widetilde{\mathcal{B}}(x)\right)\hat{r}(x,x') + \left(\widetilde{\mathcal{A}}(x)\widetilde{\mathcal{B}}(x') + \widetilde{\mathcal{D}}(x')\widetilde{\mathcal{B}}(x)\right)\hat{s}(x,x'),\end{aligned}$$

where $\hat{r}(x,x')$ is $r(x,x')$ without the matrix factor η, and lastly

$$\left\{\widetilde{\mathcal{B}}(x),\widetilde{\mathcal{B}}(x')\right\} = \left\{\widetilde{\Omega}_{12}(x),\widetilde{\Omega}_{12}(x')\right\} = \left\{\widetilde{\Omega}(x) \stackrel{\otimes}{,} \widetilde{\Omega}(x')\right\}_{12,12} = 0.$$

D. Dirac brackets of the action-angle variables

In order to isolate the action-angle variables, i.e. the physical degrees of freedom of the string, we imposed the Virasoro constraints and static gauge fixing condition on the reconstructed current. However, these constraints together form a set of second class constraints. Therefore the algebra of the action-angle variables should be expressed in terms of Dirac brackets instead of Poisson brackets. In this appendix we show that the Dirac brackets of the action-angle variables are (weakly[10]) equal to their Poisson brackets.

We start with the worldsheet action for a string on $\mathbb{R} \times S^3$ in conformal gauge. It is possible to work in conformal gauge right from the outset since the worldsheet metric and its conjugate momentum form a pair of second class constraints that commutes with the Virasoro constraints. The S^3 and $\mathbb{R}$ parts of the action decouple with the equations of motion for the $\mathbb{R}$ part being

$$d * dX_0 = 0.$$

In conformal gauge this reads $\partial_+\partial_- X_0 = 0$ which admits the general solution

$$X_0(\sigma,\tau) = X_0^+(\sigma^+) + X_0^-(\sigma^-).$$

The equations of motion for the S^3 part $d * j = 0, dj - j \wedge j = 0$ or equivalently

$$\partial_- j_+ = -\partial_+ j_- = -\frac{1}{2}[j_+, j_-], \tag{D.1}$$

can be rewritten as a zero curvature condition for a Lax connection

$$dJ(x) - J(x) \wedge J(x) = 0, \qquad J(x) = \frac{j - x * j}{1 - x^2} \in \mathfrak{sl}(2,\mathbb{C}). \tag{D.2}$$

[10]In the context of constrained Hamiltonian systems, two functions on phase-space are said to be 'weakly' equal if they are equal on the constraint surface. Note that this concept of weakness bears no relation to the notion of a 'weak' Poisson bracket introduced in section 2.4.2.

Using this flat connection we can define an algebraic curve Σ in $\mathbb{C}^2$ as a desingularisation of the spectral curve

$$\Gamma: \quad \Gamma(x,y) = \det\left(y\mathbf{1} - \Omega(x,\sigma,\tau)\right) = 0, \qquad \Omega(x,\sigma,\tau) \equiv P\overleftarrow{\exp} \int_{[c(\sigma,\tau)]} J(x) \in SL(2,\mathbb{C}).$$

As in section 3.1.3 the general solution is reconstructed by identifying the analytic properties of the Baker-Akhiezer vector $\boldsymbol{\psi}(P), P \in \Sigma$ which solves the auxiliary linear system for which (D.2) is the consistency condition

$$(d - J(x))\,\boldsymbol{\psi} = 0. \tag{D.3}$$

In order to compute the Dirac brackets one must relax the Virasoro constraints and static gauge fixing condition in the reconstruction. One then finds that $\boldsymbol{\psi}$ is uniquely determined by

$$(\psi_1) \geq \hat{\gamma}^{-1}\infty^-, \quad \psi_1(\infty^+) = 1, \quad \text{and} \quad (\psi_2) \geq \hat{\gamma}^{-1}\infty^+, \quad \psi_2(\infty^-) = 1,$$
$$\text{with} \quad \begin{cases} \psi_i(x^\pm,\sigma,\tau)\exp\left(\mp\frac{f_+(\sigma^+)}{1-x}\right) = O(1), & \text{as } x \to 1, \\ \psi_i(x^\pm,\sigma,\tau)\exp\left(\mp\frac{f_-(\sigma^-)}{1+x}\right) = O(1), & \text{as } x \to -1, \end{cases}$$

where $f_\pm$ are two arbitrary functions related to the conformal invariance of the equations of motion (D.1). Explicit reconstruction requires the introduction of an Abelian differential $d\mathcal{Q}$ of the second kind on Σ defined by its pole structure at $x = \pm 1$, namely

$$d\mathcal{Q}(x^\pm) \underset{x\to+1}{\sim} \pm f_+(\sigma^+)\frac{dx}{(1-x)^2}, \qquad d\mathcal{Q}(x^\pm) \underset{x\to-1}{\sim} \pm f_-(\sigma^-)\frac{dx}{(1+x)^2}.$$

We can write $d\mathcal{Q} = f_+(\sigma^+)dp_+ + f_-(\sigma^-)dp_-$ where $dp_\pm$ are Abelian differentials of the second kind defined by their respective poles at $x = \pm 1$,

$$dp_+(x^\pm) \underset{x\to+1}{\sim} \pm\frac{dx}{(1-x)^2}, \quad dp_-(x^\pm) \underset{x\to-1}{\sim} \pm\frac{dx}{(1+x)^2}, \quad dp_\pm \underset{x\to\mp1}{\sim} O\left((1\pm x)^0\right).$$

Just as in the flat space case, here the general solution to the equations of motion is a function of $\sigma^\pm$, through the differential $d\mathcal{Q} = f_+(\sigma^+)dp_+ + f_-(\sigma^-)dp_-$, which is what we expect since the equations of motion for the current j are conformally invariant, being derived from a conformally invariant action. So we have the following general solution for the sting moving on $\mathbb{R}\times S^3$ in conformal gauge

$$X_0^{\text{sol}}(\sigma,\tau) = X_0^+(\sigma^+) + X_0^-(\sigma^-) \in \mathbb{R}, \qquad j^{\text{sol}}(\sigma,\tau) = j\left(f_+(\sigma^+), f_-(\sigma^-)\right) \in SU(2), \tag{D.4}$$

where $X_0^\pm, f_\pm$ are arbitrary functions. We note here that the effect of the Virasoro constraint is to relate these arbitrary functions, precisely we have

$$\frac{1}{2}\text{tr} j_\pm^2 + (\partial_\pm X_0)^2 = 0 \quad \Leftrightarrow \quad f_\pm(\sigma^\pm) = X_0^\pm(\sigma^\pm).$$

The effect of the static gauge fixing condition on the other hand is to fix completely the arbitrariness of the functions $X_0^\pm$, namely

$$X_0 = \kappa\tau \quad \Leftrightarrow \quad X_0^\pm(\sigma^\pm) = \frac{\kappa}{2}\sigma^\pm.$$

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We make use of the general solutions (D.4) to parameterise the phase space variables as follows

$$(X_0(\sigma), \Pi_0(\sigma), j_\pm(\sigma)) = \left(X_0^{\rm sol}(\sigma,0), \partial_\tau X_0^{\rm sol}(\sigma,0), j_\pm^{\rm sol}(\sigma,0)\right).$$

We now need to impose the Virasoro constraints on phase-space

$$T_{\pm\pm} \equiv \frac{1}{2}{\rm tr} j_\pm^2 + \left(\frac{2\pi}{\sqrt{\lambda}}\Pi_0 \mp \partial_\sigma X_0\right)^2 \approx 0, \tag{D.5a}$$

as well as get rid of the residual gauge (i.e. conformal) invariance by imposing a further gauge fixing condition, which we choose to be the static gauge[11]

$$X_0 \approx -\frac{p_0}{\sqrt{\lambda}}\tau, \quad \Pi_0 \approx \frac{p_0}{2\pi}, \tag{D.5b}$$

where p_0 is the zero mode of Π_0. One can show that

$$\left\{\frac{1}{2}{\rm tr} j_\pm^2(\sigma), \frac{1}{2}{\rm tr} j_\pm^2(\sigma')\right\} = \pm\frac{8\pi}{\sqrt{\lambda}}\left[\frac{1}{2}{\rm tr} j_\pm^2(\sigma) + \frac{1}{2}{\rm tr} j_\pm^2(\sigma')\right]\delta'(\sigma-\sigma'). \tag{D.6}$$

and likewise $(\frac{2\pi}{\sqrt{\lambda}}\Pi_0 \mp \partial_\sigma X_0)^2$ satisfies the same equation, so that the Virasoro constraints $T_{\pm\pm}$ by themselves are first class. However, the static gauge constraints fail to commute with these and among themselves (since $\{\Pi_0(\sigma), X_0(\sigma')\} = \delta(\sigma-\sigma') \not\approx 0$), and so the constraints in (D.5) are second class. In terms of modes, the constraints (D.5) read

$$\alpha_n \approx \tilde{\alpha}_n \approx 0, \quad x_0 + \frac{p_0}{\sqrt{\lambda}}\tau \approx 0, \qquad L_n \approx \tilde{L}_n \approx 0, \quad L_0 \approx \tilde{L}_0 \approx -\frac{p_0^2}{4\sqrt{\lambda}}, \quad n \neq 0,$$

where $L_n, \tilde{L}_n$ are the fourier modes of $\frac{1}{2}{\rm tr} j_\pm^2$ respectively,

$$L_n = \frac{\sqrt{\lambda}}{8\pi}\int_0^{2\pi} d\sigma e^{in\sigma}\frac{1}{2}{\rm tr} j_+^2(\sigma), \qquad \tilde{L}_n = \frac{\sqrt{\lambda}}{8\pi}\int_0^{2\pi} d\sigma e^{-in\sigma}\frac{1}{2}{\rm tr} j_-^2(\sigma), \tag{D.7}$$

satisfying the following algebra,

$$\begin{aligned}
\{L_m, L_n\} &= i(n-m)L_{m+n},\\
\{L_m, \tilde{L}_n\} &= 0,\\
\{\tilde{L}_m, \tilde{L}_n\} &= i(n-m)\tilde{L}_{m+n},
\end{aligned}$$

which follows from (D.6), and $\alpha_n, \tilde{\alpha}_n$ are the modes of X_0 and Π_0 defined by

$$\begin{aligned}
\alpha_n &= \frac{\lambda^{\frac{1}{4}}}{\sqrt{2\pi}}\int_0^{2\pi} d\sigma e^{-in\sigma}\frac{1}{2}\left(-\frac{2\pi}{\sqrt{\lambda}}\Pi_0(\sigma) - \partial_\sigma X_0(\sigma)\right), \quad n \neq 0\\
\tilde{\alpha}_n &= \frac{\lambda^{\frac{1}{4}}}{\sqrt{2\pi}}\int_0^{2\pi} d\sigma e^{in\sigma}\frac{1}{2}\left(-\frac{2\pi}{\sqrt{\lambda}}\Pi_0(\sigma) + \partial_\sigma X_0(\sigma)\right), \quad n \neq 0\\
x_0 &= \int_0^{2\pi} d\sigma X_0(\sigma), \qquad p_0 = \int_0^{2\pi} d\sigma \Pi_0(\sigma),
\end{aligned} \tag{D.8}$$

[11] One can use the residual gauge freedom $\sigma^\pm \to \tilde{\sigma}^\pm = h_\pm(\sigma^\pm)$ to set $\tilde{\tau} \propto X_0$ since $\tilde{\tau} = \frac{1}{2}\left(h_+(\sigma^+) + h_-(\sigma^-)\right)$ solves the equations of motion for X_0. The coefficient of proportionality is forced on us by conformal invariance of the quantity $p_0 = \int_0^{2\pi} d\sigma \Pi_0(\sigma,\tau) = -\frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma \dot{X}_0(\sigma,\tau)$, so that $X_0 = -\frac{p_0}{\sqrt{\lambda}}\tilde{\tau}$.

satisfying the following algebra,

$$
\{\alpha_m, \alpha_n\} = im\delta_{m+n}, \quad \{\alpha_m, \tilde{\alpha}_n\} = 0,
$$
$$
\{\tilde{\alpha}_m, \tilde{\alpha}_n\} = im\delta_{m+n}, \quad \{p_0, x_0\} = 1.
$$

For the closed string, static gauge does not completely fix the residual gauge invariance as there still remains the possibility of rigid translations $\sigma \to \sigma + b$, which is generated by $L_0 - \tilde{L}_0$. This rigid transformation can be dealt with by symplectic reduction as explained in [1], which consists in imposing the constraint that the total worldsheet momentum vanishes $\mathcal{P} \propto L_0 - \tilde{L}_0 \propto -\sum_{I=1}^{K} n_I \mathcal{S}_I = 0$ as well as identifying points related by translations in σ. So setting aside this rigid transformation, the set of relevant constraints thus reads

$$
\alpha_n \approx \tilde{\alpha}_n \approx 0, \quad x_0 + \frac{p_0}{\sqrt{\lambda}}\tau \approx 0, \qquad L_n \approx \tilde{L}_n \approx 0, \quad L_0 + \tilde{L}_0 + \frac{p_0^2}{2\sqrt{\lambda}} \approx 0, \quad n \neq 0. \quad \text{(D.9)}
$$

In order to fix these constraints one must replace the Poisson bracket by the Dirac bracket for this set of second class constraints. The question now is whether the Dirac bracket for the action angle variables are the same as their Poisson brackets.

Now for functions f, g of the principal chiral fields j (which are independent of X_0, Π_0 and therefore commute with the constraints $\alpha_n, \tilde{\alpha}_n, x_0 + p_0\tau/\sqrt{\lambda}$) the Dirac bracket takes the schematic form

$$
\begin{aligned}
\{f,g\}_D = \{f,g\} &+ \{f,L_n\}A_{nm}\{L_m,g\} + \{f,L_0+\tilde{L}_0\}B_m\{L_m,g\} + \{f,L_m\}C_m\{L_0+\tilde{L}_0,g\} \\
&+ \{f,\tilde{L}_n\}\tilde{A}_{nm}\{\tilde{L}_m,g\} + \{f,L_0+\tilde{L}_0\}\tilde{B}_m\{\tilde{L}_m,g\} + \{f,\tilde{L}_m\}\tilde{C}_m\{L_0+\tilde{L}_0,g\}.
\end{aligned}
$$

Note that there is no term of the form "$\{f, L_0 + \tilde{L}_0\}D\{L_0 + \tilde{L}_0, g\}$" because the corresponding component D in the inverse matrix $\mathcal{C}^{-1}_{ab} = \{\phi_a, \phi_b\}^{-1}$ of the Poisson bracket of constraints vanishes. This property boils down to the fact that the constraint $x_0 + p_0\tau/\sqrt{\lambda}$ commutes with every constraint in (D.9) including itself but only fails to commute with the constraint $L_0 + \tilde{L}_0 + p_0^2/(2\sqrt{\lambda})$,

$$
\mathcal{C}_{ab} = \{\phi_a, \phi_b\} =
\begin{array}{cl}
\left(\begin{array}{c|ccccc}
0 & * & * & 0 & 0 & * \\
\hline
* & * & 0 & 0 & 0 & 0 \\
* & 0 & * & 0 & 0 & 0 \\
0 & 0 & 0 & * & 0 & 0 \\
0 & 0 & 0 & 0 & * & 0 \\
* & 0 & 0 & 0 & 0 & 0
\end{array}\right)
&
\begin{array}{l}
L_0 + \tilde{L}_0 + p_0^2/(2\sqrt{\lambda}) \\
L_n \\
\tilde{L}_n \\
\alpha_n \\
\tilde{\alpha}_n \\
x_0 + p_0\tau/\sqrt{\lambda}
\end{array}
\end{array}
.
$$

It follows that for functions f, g of the principal chiral model that are invariant under residual gauge transformations generated by $L_n, \tilde{L}_n, n \neq 0$ we have the desired equality of Dirac and Poisson brackets

$$
\{f, g\}_D = \{f, g\}.
$$

It therefore remains to check that the action angle variables can be defined in a conformally invariant way from the general solution $j^{\text{sol}}(\sigma, \tau)$ obtained in (D.4).

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Going back to expression (D.4) we see that the periodicity requirement of the solution under $\sigma \to \sigma + 2\pi$ leads to the conditions

$$[f_\pm(\sigma + 2\pi) - f_\pm(\sigma)] \int_{\boldsymbol{b}} dp_\pm \in 2\pi\mathbb{Z}^g + 2\pi\Pi\mathbb{Z}^g.$$

Since the Jacobian lattice $2\pi\mathbb{Z}^g + 2\pi\Pi\mathbb{Z}^g$ is discrete, this in turn requires the expression in square brackets to be constant, i.e. independent of σ, say $f_\pm(\sigma + 2\pi) - f_\pm(\sigma) = 2\pi k^\pm$, $k^\pm \in \mathbb{C}$. Then $f_\pm(\sigma) - k^\pm\sigma$ is periodic under $\sigma \to \sigma + 2\pi$, which means that we can decompose the functions $f_\pm$ as follows

$$f_\pm(\sigma) = \xi_0^\pm + k^\pm\sigma + \sum_{n\neq 0} \xi_n^\pm e^{in\sigma}. \tag{D.10}$$

Recall that imposing the Virasoro constraint on the solution $j^{\text{sol}}(\sigma,\tau)$ has the effect of rendering the functions $f_\pm$ linear, and so this corresponds to setting all the modes $\xi_n^\pm$ in (D.10) to zero, i.e.

$$\frac{1}{2}\text{tr}\left(j_\pm^{\text{sol}}\right)^2 = -\frac{p_0^2}{\lambda} \qquad \Leftrightarrow \qquad \xi_n^\pm = 0,\ \forall n, \quad k^\pm = \frac{ip_0}{2\sqrt{\lambda}}. \tag{D.11}$$

As we now show, the effect of the Virasoro constraints on the functions $f_\pm$ can be deduced from the following brackets

$$\begin{aligned}
\left\{\frac{\sqrt{\lambda}}{4\pi}\frac{1}{2}\text{tr} j_\pm^2(\sigma), j_\pm^b(\sigma')\right\} &= \frac{1}{2}\left[j_\pm(\sigma), j_\mp(\sigma)\right]^b \delta(\sigma-\sigma') \pm 2 j_\pm^b(\sigma)\delta'(\sigma-\sigma'),\\
\left\{\frac{\sqrt{\lambda}}{4\pi}\frac{1}{2}\text{tr} j_\pm^2(\sigma), j_\mp^b(\sigma')\right\} &= -\frac{1}{2}\left[j_\pm(\sigma), j_\mp(\sigma)\right]^b \delta(\sigma-\sigma'),
\end{aligned}$$

which are a consequence of the non-ultra local brackets of the principal chiral model. Let $j^{\text{sol}}(\sigma,\tau)$ be a physical path, i.e. satisfying the equations of motion (D.1), then one can deduce immediately from the above brackets that

$$\begin{aligned}
\left\{\frac{\sqrt{\lambda}}{4\pi}\int \epsilon^\pm(\sigma'\pm\tau)\frac{1}{2}\text{tr} j_\pm^2(\sigma')d\sigma', j_\pm^b(\sigma)\right\}\left(j^{\text{sol}}(\sigma,\tau)\right) &= -\partial_\pm\left(\epsilon^\pm(\sigma\pm\tau) j_\pm^{\text{sol}}(\sigma,\tau)\right)^b,\\
\left\{\frac{\sqrt{\lambda}}{4\pi}\int \epsilon^\pm(\sigma'\pm\tau)\frac{1}{2}\text{tr} j_\pm^2(\sigma')d\sigma', j_\mp^b(\sigma)\right\}\left(j^{\text{sol}}(\sigma,\tau)\right) &= -\epsilon^\pm(\sigma\pm\tau)\left(\partial_\pm j_\mp^{\text{sol}}(\sigma,\tau)\right)^b.
\end{aligned}$$

So $j_\pm^b$ transforms as a scalar under $\frac{1}{2}\text{tr} j_\mp^2$ but as a scalar density of weight 1 under $\frac{1}{2}\text{tr} j_\pm^2$ (this is in agreement with the fact that the Langrangian $L \propto \text{tr}(j_+ j_-)$ should be a density of weight 1 under coordinate transformations). Because $j^{\text{sol}}(\sigma,\tau) = j(f_+(\sigma^+), f_-(\sigma^-))$ one can now derive the action of $L_n, \tilde{L}_n$ on the functions $f_\pm$, namely

$$\begin{aligned}
\{L_n, f_+(\sigma^+)\} &= -e^{in\sigma^+}\partial_{\sigma^+} f_+(\sigma^+), & \{L_n, f_-(\sigma^-)\} &= -e^{in\sigma^+}\partial_{\sigma^+} f_-(\sigma^-) = 0,\\
\{\tilde{L}_n, f_+(\sigma^+)\} &= -e^{-in\sigma^-}\partial_{\sigma^-} f_+(\sigma^+) = 0, & \{\tilde{L}_n, f_-(\sigma^-)\} &= -e^{-in\sigma^-}\partial_{\sigma^-} f_-(\sigma^-).
\end{aligned}$$

Using the definition (D.10) of the functions $f_\pm$ we are now able to write the action of the Virasoro constraints $L_n, \tilde{L}_n, n \neq 0$ on the parameters ξ^+_m and ξ^-_m. Explicitly we find

$$\{L_n, \xi^-_m\} = 0, \qquad \{L_n, \xi^+_m\} = \delta_{mn} k^+ - (m-n)\xi^+_{m-n},$$
$$\{\tilde{L}_n, \xi^+_m\} = 0, \qquad \{\tilde{L}_n, \xi^-_m\} = \delta_{mn} k^- - (m-n)\xi^-_{m-n}.$$

In particular, on the constraint surface (D.11) we get

$$\{L_n, \xi^+_m\} \approx \delta_{mn} k^+, \qquad \{\tilde{L}_n, \xi^-_m\} \approx \delta_{mn} k^-, \quad n \neq 0. \tag{D.12}$$

Recalling how the σ, τ evolution of the solution is expressed in terms of the Abelian differential of the second kind $d\mathcal{Q} = f_+(\sigma^+) dp_+ + f_-(\sigma^-) dp_-$ as

$$\boldsymbol{\theta}(\sigma,\tau) = \boldsymbol{\theta}_0 - \int_{\boldsymbol{b}} d\mathcal{Q} = \boldsymbol{\theta}_0 - f_+(\sigma^+) \int_{\boldsymbol{b}} dp_+ - f_-(\sigma^-) \int_{\boldsymbol{b}} dp_-,$$
$$\bar{\theta}(\sigma,\tau) = \bar{\theta}_0 - \int_{\infty^-}^{\infty^+} d\mathcal{Q} = \bar{\theta}_0 - f_+(\sigma^+) \int_{\infty^-}^{\infty^+} dp_+ - f_-(\sigma^-) \int_{\infty^-}^{\infty^+} dp_-,$$

the angle variables were defined in [1] and in section 3.4 simply as the parameters $\boldsymbol{\theta}(0,0), \bar{\theta}(0,0)$. A more suitable definition here, valid off the constraint surface, would be instead to take the angle variable $\varphi_I, I = 1, \ldots, K = g+1$ as the zero mode of $\theta_i(\sigma, 0), \bar{\theta}(\sigma, 0)$, namely on $J(\Sigma) \times \mathbb{C}/2\pi\mathbb{Z}$ we define

$$\begin{aligned} \boldsymbol{\varphi} &= \boldsymbol{\theta}_0 - \xi^+_0 \int_{\boldsymbol{b}} dp_+ - \xi^-_0 \int_{\boldsymbol{b}} dp_-, \\ \varphi_K &= \bar{\theta}_0 - \xi^+_0 \int_{\infty^-}^{\infty^+} dp_+ - \xi^-_0 \int_{\infty^-}^{\infty^+} dp_-. \end{aligned} \tag{D.13}$$

The difference between these two definitions is the following vector in $J(\Sigma) \times \mathbb{C}/2\pi\mathbb{Z}$

$$\boldsymbol{\theta}(0,0) - \boldsymbol{\varphi} = -\left(\sum_{n\neq 0} \xi^+_n\right) \int_{\boldsymbol{b}} dp_+ - \left(\sum_{n\neq 0} \xi^-_n\right) \int_{\boldsymbol{b}} dp_- \approx 0,$$
$$\bar{\theta}(0,0) - \varphi_K = -\left(\sum_{n\neq 0} \xi^+_n\right) \int_{\infty^-}^{\infty^+} dp_+ - \left(\sum_{n\neq 0} \xi^-_n\right) \int_{\infty^-}^{\infty^+} dp_- \approx 0,$$

which vanishes on the constraint surface. In particular, on the constraint surface (D.11) we have by (D.12) and (D.13)

$$\{L_n, \varphi_I\} \approx \{\tilde{L}_n, \varphi_I\} \approx 0, \quad I = 1, \ldots, K.$$

Since the action variables $S_I, I = 1, \ldots, K$ are invariant under σ, τ evolution, they obviously Poisson commute with the generators of conformal transformation $L_n, \tilde{L}_n$ and so we also have

$$\{L_n, S_I\} = \{\tilde{L}_n, S_I\} = 0, \quad I = 1, \ldots, K.$$

So finally we have established equality of the Dirac and Poisson bracket of the action angle variables on the constraint surface,

$$\{f, g\}_D \approx \{f, g\}, \qquad \text{for } f, g \in \{\varphi_I, S_I\}_{I=1}^K.$$

References

[1] N. Dorey and B. Vicedo, *On the dynamics of finite-gap solutions in classical string theory*, *JHEP* **07** (2006) 014 [hep-th/0601194].

[2] L.D. Faddeev and L.A. Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer-Verlag Berlin (1987).

[3] I. Krichever, *Vector bundles and Lax equations on algebraic curves*, *Commun. Math. Phys.* **229** (2002) 229 [hep-th/0108110].

[4] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, *Nucl. Phys.* **B 533** (1998) 109 [hep-th/9805028].

[5] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev.* **D 69** (2004) 046002 [hep-th/0305116].

[6] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical/quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024 [hep-th/0402207].

[7] N. Beisert, V.A. Kazakov and K. Sakai, *Algebraic curve for the* SO(6) *sector of AdS/CFT*, *Commun. Math. Phys.* **263** (2006) 611 [hep-th/0410253].

[8] N. Beisert, V.A. Kazakov, K. Sakai and K. Zarembo, *The algebraic curve of classical superstrings on $AdS_5 \times S^5$*, *Commun. Math. Phys.* **263** (2006) 659 [hep-th/0502226].

[9] O. Babelon, D. Bernard, M. Talon, *Introduction to classical integrable systems*, Cambridge University Press (2003).

[10] M.B. Green, J.H. Schwarz and E. Witten, *Superstring theory. Vol. 1: introduction*, Cambridge University Press.

[11] L. Brink and M. Henneaux, *Principles of string theory*, Plenum Press, New York (1988).

[12] I.M. Krichever, *Integration of non-linear equations by methods of algebraic geometry*, *Funct. Anal. Appl.* **11** (1977) 12.

[13] I.M. Krichever, *Methods of algebraic geometry in the theory of non-linear equations*, *Russian Math. Surveys* **32** (1977) 185.

[14] I.M. Krichever and D.H. Phong, *On the integrable geometry of soliton equations and $N = 2$ supersymmetric gauge theories*, *J. Diff. Geom.* **45** (1997) 349 [hep-th/9604199].

[15] I.M. Krichever, *Two-dimensional algebraic-geometrical operators with self-consistent potentials*, *Func. An & Apps.* **28** (1994) 26.

[16] E.D. Belokolos, A.I. Bobenko, V.Z. Enol'skii, A.R. Its, V.B. Matveev, *Algebro-geometric approach to nonlinear integrable equations*, Springer-Verlag Telos (1994).

[17] E.K. Sklyanin, *Separation of variables - new trends*, *Prog. Theor. Phys. Suppl.* **118** (1995) 35.

[18] J.M. Maillet, *Kac-Moody algebra and extended Yang-Baxter relations in the $O(N)$ nonlinear σ-model*, *Phys. Lett.* **B 162** (1985) 137.

[19] J.M. Maillet, *Hamiltonian structures for integrable classical theories from graded Kac-Moody algebras*, *Phys. Lett.* **B 167** (1986) 401.

[20] J.M. Maillet, *New integrable canonical structures in two-dimensional models*, *Nucl. Phys.* **B 269** (1986) 54.

[21] M. Forger, M. Bordemann, J. Laartz and U. Schaper, *The Lie-Poisson structure of integrable classical nonlinear sigma models*, *Commun. Math. Phys.* **152** (1993) 167 [hep-th/9201051].

[22] Y. Fedorov, *Classical integrable systems and billiards related to generalized jacobians*, *Acta Appl. Math.* **55** (1999) 151.